Math 116 Spring 08

Selected solutions to Exercises 2.4.

9. Suppose that $a | c$ and $b | c$, and $(a,b) = 1$.

Then $a | c \Rightarrow c = aq$ for some $q \in \mathbb{Z}$.

Also $b | c \Rightarrow b | aq \Rightarrow b | q$ (by Th. 2.14 since $(a,b) = 1$)

$\Rightarrow q = br$ for some $r \in \mathbb{Z}$

ie $c = aq = abr \Rightarrow ab | c$.

19. Suppose that $d = (a,b)$, $a | c$ and $b | c$.

Then $a = a'd$, $b = b'd$ and $(a', b') = 1$

Since if $(a', b') = d'$ with $d > 1$

then $d' | a' \Rightarrow d'd | a'd$ ie $d'd | a$

similarly $d'd | b$. Thus $d'd | d$.

But $d | d'd$ and both $d > 0$ and $d'd > 0$.

Thus $d'd = d$ ie $d' = 1$. This contradiction shows that $(a', b') = 1$.

Now $a | c \Rightarrow c = aq = a'dq$ for some $q \in \mathbb{Z}$.

Also $b | c \Rightarrow b | a'dq \Rightarrow b'd | a'dq \Rightarrow b' | a'q$

$\Rightarrow b' | q$ (by Th. 2.14 since $(a', b') = 1$)

Thus $q = b'r$ and $c = aq = ab'r$

Thus $cd = ab'dr = abr \Rightarrow ab | cd$.

20. Let $S = \{ n \mid a | n \text{ and } b | n \}$ and let S^+ be the set of all positive integers in S.

Then S contains $\pm ab \Rightarrow S^+ \neq \emptyset$.

Let m be the least element of S^+. Then $m > 0$ and $a | m$ and $b | m$. We show that if $a | c$ and $b | c$, then $m | c$.

So suppose that c is such that $a|c$ and $b|c$.

Let $d = (m, c)$. Then $d > 0$ and

$d = mp + cq$ for some $p, q \in \mathbb{Z}$

Thus $a|d$ and $b|d$. Thus $d \in S^+$.

Also $d | m \Rightarrow m = dr$ for some $r \geqslant 1$

So $d \leqslant m$. But m is least element of $S^+ \Rightarrow d = m$

Thus $m = (m, c) \Rightarrow m|c$.

21. Suppose that $d = (a,b)$, $a = a_0 d$ and $b = b_0 d$ >0 >0

Let $m = a_0 b_0 d$. Then $m > 0$ and $a|m$, $b|m$.

Suppose $a|c$ and $b|c$.

Then $a|c \Rightarrow c = ap = a_0 dp$ for some $p \in \mathbb{Z}$

Also $b|c \Rightarrow b_0 d | a_0 dp \Rightarrow b_0 | a_0 p$.

But since $(a_0, b_0) = 1$ (see solution to q. 19 above)

Th. 2.14 $\Rightarrow b_0 | p \Rightarrow p = b_0 q \Rightarrow c = a_0 d b_0 q$

$= mq$. Thus $m|c$.

27. Suppose that $a^2 = 2b^2$ and $(a,b) = d > 1$

Then $a = a_0 d$, $b = b_0 d$, $(a_0, b_0) = 1$

and $a_0^2 d^2 = 2 b_0^2 d^2$ ie $a_0^2 = 2b_0^2$.

Thus we can assume that $(a,b) = 1$.

Now $a^2 = 2b^2 \Rightarrow 2 | a^2$ ie $2 | a.a$.

By Theorem 2.16 $2|a$ ie $a = 2p$.

Thus $a^2 = 2b^2 \Rightarrow 4p^2 = 2b^2 \Rightarrow 2p^2 = b^2$

So $2 | b^2$ & $2|b$.

Thus $2|a$ and $2|b \Rightarrow 2 | (a,b)$.

But we are assuming $(a,b) = 1$, a contradiction. Thus

a and b cannot exist.

Suppose that $\sqrt{2} \in \mathbb{Q}$.

Then $\sqrt{2} = \dfrac{a}{b}$ for integers a, b

$\Rightarrow \quad a^2 = 2b^2$.

But we have just seen that $a^2 = 2b^2$ has no solution in integers. Thus $\sqrt{2} \notin \mathbb{Q}$.