(I)

Math 116, Spring 2008

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EXERCISES 4.2

3) Let $f_a : G \to G$, $[x] \mapsto [a][x]$ where $G = \{[2],[4],[6],[8]\} \subseteq \mathbb{Z}_{10}$.

Then

$$f_2 : \begin{cases} f_2([2]) = [4] \\ f_2([4]) = [8] \\ f_2([6]) = [2] \\ f_2([8]) = [6] \end{cases} \quad f_4 : \begin{cases} f_4([2]) = [8] \\ f_4([4]) = [6] \\ f_4([6]) = [4] \\ f_4([8]) = [2] \end{cases} \quad f_6 : \begin{cases} f_6([2]) = [2] \\ f_6([4]) = [4] \\ f_6([6]) = [6] \\ f_6([8]) = [8] \end{cases} \quad f_8 : \begin{cases} f_8([2]) = [6] \\ f_8([4]) = [2] \\ f_8([6]) = [8] \\ f_8([8]) = [4] \end{cases}$$

The set $G' = \{f_2, f_4, f_6, f_8\}$ is a group of permutations.

Let $\phi : G \to G'$, $[x] \mapsto f_x$. ϕ is an isomorphism from G to G'.

(In fact;

$G' \cong \{(1), (1,4)(2,3), (1,3,4,2), (1,2,4,3)\} \leqslant S_4$)

6) Let G be a group and $k_a : G \to G$, $x \mapsto xa^{-1}$, $a \in G$.

(a) k_a is one-to-one:

$k_a(x) = k_a(y) \Rightarrow xa^{-1} = ya^{-1} \Rightarrow x = y$ where $x, y \in G$.

k_a is onto:

Let $x \in G$. Since G is a group and $a, x \in G$, $xa \in G$.

Then $k_a(xa) = xaa^{-1} = x$.

So k_a is a permutation on the set of elements in G.

(b) $K = \{k_a \mid a \in G\}$

i) $(k_a \circ k_b)(x) = xb^{-1}a^{-1} = k_{ab} \in K$ since $ab \in G$.

So K is closed under composition.

ii) $((k_a \circ k_b) \circ k_c)(x) = (k_a \circ k_b)(xc^{-1})$
$= xc^{-1}b^{-1}a^{-1}$
$= k_a(xc^{-1}b^{-1})$
$= k_a \circ (k_b \circ k_c)$

So composition is associative.

(iii) Let e be identity element of G. Then

$(k_a \circ k_e)(x) = k_a(xe^{-1}) = xa^{-1} = k_a(x) = xa^{-1}e^{-1} = (k_e \circ k_a)(x)$

So k_e is identity element of K

(iv) Let $k_a \in K$. Then

$$\begin{aligned}(k_a \circ k_{a^{-1}})(x) &= x(a^{-1})^{-1}a^{-1} \\ &= xe \\ &= k_e(x)\end{aligned}$$

$$\begin{aligned}(k_{a^{-1}} \circ k_a)(x) &= x\,a^{-1}(a^{-1})^{-1} \\ &= xe \\ &= k_e(x)\end{aligned}$$

So $k_{a^{-1}}$ is the inverse of k_a.

Therefore K is a group wrt mapping composition

(c) Let $\emptyset : G \longrightarrow K$, $a \longmapsto k_a$

(i) Let $a, b \in G$. Then

$$\begin{aligned}\emptyset(ab) &= k_{ab} \\ &= k_a \circ k_b \\ &= \emptyset(a) \circ \emptyset(b)\end{aligned}$$

So $\emptyset$ is homeomorphism

(ii) $$\begin{aligned}\emptyset(a) = \emptyset(b) &\Longrightarrow k_a = k_b \\ &\Longrightarrow xa^{-1} = xb^{-1}, \quad \forall x \in G \\ &\Longrightarrow a^{-1} = b^{-1} \\ &\Longrightarrow a = b\end{aligned}$$

So $\emptyset$ is one-to-one

(iii) Let $k_a \in K$. Then $a \in G$ st

$\emptyset(a) = k_a$

So $\emptyset$ is onto.

Therefore $\emptyset$ is an isomorphism.

4.4. 8). $H \leq G \Rightarrow gHg^{-1} \leq G \ \forall g \in G$.

$e = geg^{-1} \in gHg^{-1}$. So $gHg^{-1} \neq \phi$.

Let $x, y \in gHg^{-1}$.

Then $x = gh_1g^{-1}$, $y = gh_2g^{-1}$ for some $h_1, h_2 \in H$.

Hence $x \cdot y^{-1} = gh_1g^{-1} \cdot (gh_2g^{-1})^{-1} = gh_1g^{-1}(g^{-1})^{-1} \cdot h_2^{-1} g^{-1}$
$= gh_1g^{-1}gh_2^{-1}g^{-1}$
$= g(h_1h_2^{-1})g^{-1} \in gHg^{-1}$.

Therefore $gHg^{-1} \leq G$.

4.4.25) Claim: $H \leq G$, $K \trianglelefteq G \Rightarrow H \cdot K = KH$.

Let $x \in HK$. Then $x = h \cdot k$ for some $h \in H$, for some $k \in K$.

Since $x = h \cdot k$, and $k \in K$, $x = hk \in hK$.

Since $K \trianglelefteq G$, $hK = Kh$. Hence $x \in Kh$.

Since $Kh \subseteq KH$ & $x \in Kh$, $x \in KH$. Thus $HK \subseteq KH$.

Similarly, for $y \in KH$, $y = k \cdot h$ for some $h \in H, k \in K$.

Hence $y = kh \in Kh \underset{\uparrow}{=} hK \subseteq HK$. Thus $KH \subseteq HK$.

as $K \trianglelefteq G$

Therefore $HK = KH$.

26) If $H \leq G$, $K \trianglelefteq G$, then $HK \leq G$.

Let $x, y \in HK$. Then $x = h_1k_1$, $y = h_2k_2$ for some $h_1, h_2 \in H$

$k_1, k_2 \in K$. Then $x \cdot y^{-1} = h_1\underbrace{k_1 k_2^{-1}}_{= k_3} h_2^{-1} = h_1\underbrace{(k_3 h_2^{-1})}_{\in KH = HK} = h_1(h_3 k_4) \in HK$

where $h_3 \in H$, $k_4 \in K$, s.t. $k_3 h_2^{-1} = h_3 k_4$ (bec. $KH = HK$ by Prob. (25))