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CHAPTER 3.4

6) Let $G = \{[a]_7 \in \mathbb{Z}_7 \mid [a]_7 \neq [0]_7\}$ be the multiplicative group

Let $\mathbb{Z}_6 = \{[a]_6\}$ be the additive group.

Let $\phi : \mathbb{Z}_6 \longrightarrow G$
$[a]_6 \longmapsto ([5])^a$

(i) $\phi([a]+[b]) = ([5])^{a+b}$
$= ([5])^a([5])^b$
$= \phi([a])\,\phi([b])$

one-to-one (ii) $\phi([a]) = \phi([b]) \Rightarrow ([5])^a = ([5])^b$
$\Rightarrow a \equiv b \pmod 6$
$\Rightarrow a = b$ since $a, b \in \mathbb{Z}_6$ (so $0 \leq a, b < 6$)

onto (iii) Let $x \in G$. Then $x = ([5])^t$ for some $0 \leq t < 6$ (since G is cyclic and $[5]$ is generator)

Then $([5])^t = \phi([t])$

13) Let G be arbitrary group.
$\phi : G \to G$
$a \mapsto a^{-1}$

Claim: ϕ is not isomorphism.

Counter example is symmetric group of order 3

Let $G = S_3 = \{e, \rho, \rho^2, \alpha, \gamma, \delta\}$

Group Table

	e	ρ	ρ^2	α	γ	δ
e	e	ρ	ρ^2	α	γ	δ
ρ	ρ	ρ^2	e	γ	δ	α
ρ^2	ρ^2	e	ρ	δ	α	γ
α	α	δ	γ	e	ρ^2	ρ
γ	γ	α	δ	ρ	e	ρ^2
δ	δ	γ	α	ρ^2	ρ	e

$\phi(\alpha \circ \gamma) = (\rho^2)^{-1} = \rho$

$\phi(\alpha) \circ \phi(\gamma) = (\alpha)^{-1} \circ (\gamma)^{-1} = \alpha \circ \gamma = \rho^2$

$\phi(\alpha \circ \gamma) \neq \phi(\alpha) \circ \phi(\gamma)$ since $\rho \neq \rho^2$

$\therefore$ ϕ is not isomorphism

16) Let 'a' be arbitrary element of a group G.

Let $t_a : G \to G$
$x \longmapsto axa^{-1}$

Claim: t_a is isomorphism (infact automorphism)

(i) Let $x, y \in G$

$$t_a(xy) = axya^{-1}$$
$$= axeya^{-1}$$
$$= axa^{-1}aya^{-1} = t_a(x)\, t_a(y)$$

one-to-one (ii) $t_a(x) = t_a(y) \Rightarrow axa^{-1} = aya^{-1}$

$\Rightarrow a^{-1}axa^{-1} = a^{-1}aya^{-1}$

$\Rightarrow xa^{-1} = ya^{-1}$

$\Rightarrow xa^{-1}a = ya^{-1}a$

$\Rightarrow x = y$

onto (iii) Let $x \in G$. Then $a^{-1}xa \in G$. Then

$$t_a(a^{-1}xa) = aa^{-1}xaa^{-1}$$
$$= exe$$
$$= x$$

22) Let G be an infinite cyclic group and 'a' be a generator of G.

Then $\phi : \mathbb{Z} \to G$
$x \longmapsto a^x$

Claim: ϕ is isomorphism

(i) Let $x, y \in \mathbb{Z}$

$$\phi(x+y) = a^{x+y}$$
$$= a^x a^y$$
$$= \phi(x)\phi(y)$$

one-to-one (ii) $\phi(x) = \phi(y) \Rightarrow a^x = a^y$

$\Rightarrow x = y$ (since G is infinite)

onto (iii) Let $b = a^x \in G$. Then $\exists x \in \mathbb{Z}$ st

$$\phi(x) = a^x = b$$

23) Let G be a finite cyclic group and 'a' be a generator of G of order n.

Then $\phi : \mathbb{Z}_n \to G$
$[x] \longmapsto a^x$

Claim: ϕ is isomorphism

(i) Let $[x], [y] \in \mathbb{Z}_n$

$$\phi([x]+[y]) = a^{x+y} = a^x a^y = \phi([x]) \cdot \phi([y])$$

one-to-one (ii) $\phi([x]) = \phi([y]) \Rightarrow a^x = a^y \underset{\substack{\text{since}\\ G \text{ is finite}}}{\Rightarrow} x \equiv y \pmod{n} \underset{\substack{\text{since}\\ 0 \leq x, y < n}}{\Rightarrow} x = y$

onto (iii) Let $b = a^x \in G$ where $0 \leq x < n$. Then $\exists [x] \in \mathbb{Z}_n$ st

$$\phi([x]) = a^x = b$$