SPRING 2008, Math 116

Solutions to some exercises from 3.1 & 3.2

§3.1, p. 118, 6-) $G_n = \{x \in \mathbb{C} : x^n = 1\}$. Let $x, y \in G$. Then $x^n = 1 = y^n$, so $(xy)^n = x^n y^n = 1 \cdot 1 = 1$, and G_n is closed under multiplication. We know multiplication of complex numbers is associative. $1 \in G_n$ since $1^n = 1$. Also if $x \in G_n$, then $1 \cdot x = x \in G_n$ and $x \cdot 1 = x \in G_n$. Hence 1 is an identity element in G_n. Finally, if $x \in G_n$, then $\frac{1}{x} \in G_n$ since $\left(\frac{1}{x}\right)^n = \frac{1}{x^n} = \frac{1}{1} = 1$. Also $x \cdot \frac{1}{x} = 1$ and $\frac{1}{x} \cdot x = 1$. Thus $\frac{1}{x} = x^{-1}$ is the inverse of x in G_n. We conclude that G_n with operation multiplication is a group. Additionally, $o(G_n) = n$.

14-) Multiply both sides of $ca = a$ on the right by a^{-1}; we get $c = e =$ identity.

x	a	b	c	d
a	c	d	a	b
b	d	a	b	c
c	a	b	c	d
d	b	c	d	a

This fills the column and row of c. Next we use trial and error. <u>If</u> $ba = c$, the $ab = c$ too, and a and b are inverses. Then $b = bc = bba = b^2 a = a^2$. Then we look at ad. $ad \neq a$, $ad \neq d$ since neither a nor d is the identity. $ad \neq c$ either since $a^{-1} \neq d$. Then $ad = b$. But then $b^2 d = b$ and this gives $bd = c$ after cancelling b. This means $b^{-1} = d$. But above we said $b^{-1} = a$. Thus the <u>guess</u> $ba = c$ was <u>wrong.</u> This leaves us with the only choice $bd = c = db$, that is, d and b are inverses. Since $c^{-1} = c$ also, we must have $a^{-1} = a$ as well, that is, $a^2 = c$. The rest is easy. $ab \neq a$, $ab \neq b$ since neither a nor b is the identity. $ab \neq c$ either since $a^{-1} \neq b$. Then $ab = d$. By similar reasoning $ba = d$, $da = b$, $ad = b$. Finally $d^2 \neq d$ since d is not the identity, $d^2 \neq c$ since $d^{-1} \neq d$. If $d^2 = b$, then $a = b^2 = d^2 b = ddb = dc = d$, but a and d are distinct elements. Thus we must have $d^2 = a$.

32-) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} ad & 0 & 0 \\ 0 & be & 0 \\ 0 & 0 & cf \end{bmatrix}$, and $ad \neq 0$, $be \neq 0$, $cf \neq 0$ if $a \neq 0$, $b \neq 0$, $c \neq 0$, $d \neq 0$, $e \neq 0$, and $f \neq 0$. Thus G is closed under matrix multiplication.

We know matrix multiplication is associative. The 3×3 identity matrix I_3 is in G, and $I_3A = AI_3 = A$ for any $A\in G$. So I_3 is an identity for G. Finally, if $A=\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c\end{bmatrix}\in G$, then $B=\begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c\end{bmatrix}\in G$ too, and $AB = BA = I_3$; that is, G contains the inverse of each of its elements. Thus G is a group under multiplication.

47-) We are given $xyxy = xxyy$ for all $x,y\in G$. Multiply by y^{-1} on the right and by x^{-1} on the left on both sides. We get $yx = xy$ for all $x,y\in G$. Thus G is abelian.

§3.2, p. 129, 10-) $0\in H$ clearly, so H is nonempty. Next let $x,y\in H$. So $ax\equiv 0 \pmod n$ and $ay\equiv 0 \pmod n$. Add; by Theorem 2.23, we have $ax+ay\equiv 0 \pmod n$, that is, $a(x+y)\equiv 0 \pmod n$. Then $x+y\in H$ too. Lastly let $x\in H$. So $ax\equiv 0 \pmod n$. Multiply by -1. By Theorem 2.22, we have $-ax\equiv 0 \pmod n$, that is, $a(-x)\equiv 0 \pmod n$. Then $-x\in H$ too. Thus H is a subgroup of $\mathbb{Z}$ under addition.

13-) d-) $\begin{bmatrix} 1 & 2 \\ -3 & 0\end{bmatrix}\in H$, so H is nonempty. Now let $A=\begin{bmatrix} x & y \\ z & w\end{bmatrix}$ and B $\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ be in H. Then $x+y+z+w=0=a+b+c+d$. $A+B=\begin{bmatrix} x+a & y+b \\ z+c & w+d\end{bmatrix}$ and $(x+a)+(y+b)+(z+c)+(w+d) = (x+y+z+w)+(a+b+c+d) = 0+0=0$. So $A+B\in H$ too. Finally, $-A=\begin{bmatrix} -x & -y \\ -z & -w\end{bmatrix}\in H$ since $-x-y-z-w=-(x+y+z+w)=-0=0$. Thus H is a subgroup of 2×2 matrices with integer entries.

23-) $a\in C_a$, so $C_a\neq\emptyset$. Second, let $x,y\in C_a$. $a(xy)=(ax)y=(xa)y=x(ay)= = x(ya)=(xy)a$ since $ax=xa$ and $ay=ya$. Then $xy\in C_a$ too. Third, let $x\in C_a$. So $ax=xa$. Multiply both sides on the right by x^{-1} and on the left also by x^{-1}. We get $x^{-1}a=ax^{-1}$, that is, $x^{-1}\in C_a$ too. Thus C_a is a subgroup of G.

27-) Let $H_1=\{e,\rho,\rho^2\}$ and $H_2=\{e,\sigma\}$. Both H_1 and H_2 are subgroups of S_3. $H_1\cup H_2=\{e,\rho,\rho^2,\sigma\}$ is not a subgroup of S_3, because $\rho\sigma = \quad \notin H_1\cup H_2$.