1.4. 10. n). Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = \begin{cases} x+1 & \text{if } x \text{ is even} \\ \frac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$

Does f have a right inverse?
If yes, find one.

Let's check if f is onto.
Let $x \in \mathbb{Z}$. Then $2x-1$ is an odd integer.
Then $f(2x-1) = \frac{2x-1+1}{2} = x$. This shows that f is onto.

Hence f has a right inverse. Define $g: \mathbb{Z} \to \mathbb{Z}$ by
$g(x) = 2x-1$ for $x \in \mathbb{Z}$.
Then g is a right inverse for f, because

$f(g(x)) = f(\underset{\text{odd}}{2x-1}) = \frac{2x-1+1}{2} = x$ for all $x \in \mathbb{Z}$.

Hence $f \circ g = 1_{\mathbb{Z}}$.

Note that right inverse is not unique. Define $h: \mathbb{Z} \to \mathbb{Z}$
by $h(x) = \begin{cases} x-1 & \text{if } x \text{ is odd} \\ 2x-1 & \text{if } x \text{ is even} \end{cases}$ for all $x \in \mathbb{Z}$.

Then $f(h(x)) = f(x-1) = x-1+1 = x$ for odd $x \in \mathbb{Z}$.
& $f(h(x)) = f(2x-1) = \frac{2x-1+1}{2} = x$ for even $x \in \mathbb{Z}$.
Therefore $f \circ h = I_{\mathbb{Z}}$, hence h is another right inverse for f.

1.4. 11 n). Does the above f have a left inverse? If yes, find one.

We must check if f is one-to-one or not!
Since f is defined differently for even & odd integers,
we need to consider ~~various~~ possible cases.
Recall that f is one-to-one if the following is true.
" $f(x) = f(y)$ implies $x = y$ for all $x, y \in \mathbb{Z}$ "

Case 1. x, y are both even.
Case 2. x, y are both odd.
Case 3. x is even, y is odd. (Note that we might consider x is odd, y is even as well.)

Case 1. x, y are even.
Suppose $f(x) = f(y)$, then $x+1 = y+1$. Hence $x = y$.

Case 2. x, y are odd.
Suppose $f(x) = f(y)$, then $\frac{x+1}{2} = \frac{y+1}{2}$. Hence $x = y$.

Case 3. x is even, y is odd
Suppose $f(x) = f(y)$, then $f(x) = x+1 = \frac{y+1}{2} = f(y)$.
Then $2x+2 = y+1$, hence $y = 2x+1$.
We did not get $y = x$.
Let's take $x = 2$, and $y = 2 \cdot 2 + 1 = 5$.
Then $f(2) = 3$ and $f(5) = \frac{5+1}{2} = 3$
Therefore $f(2) = f(5)$ even if $2 \neq 5$.
Therefore f is not one-to-one.
Thus f has no left inverse.

1.4. 14). Let $f: A \to A$, where A is nonempty. Prove that f has a right inverse if and only if $f(f^{-1}(T)) = T$ for every $T \subseteq A$.

$\Leftarrow$: Suppose $f(f^{-1}(T)) = T$ for all $T \subseteq A$.
We need to show f is onto.
Let $a \in A$, we must find $x \in A$ such that $f(x) = a$.
Let $T = \{a\}$. Then by the hypothesis/assumption we have
$f(f^{-1}(T)) = T = \{a\} \neq \phi$. Hence $f^{-1}(T) \neq \phi$.
Note that $f^{-1}(T) = \{b \in A \mid f(b) \in T\} = \{b \in A \mid f(b) \in \{a\}\}$
$= \{b \in A \mid f(b) = a\} \neq \phi$.
Since $f^{-1}(T) \neq \phi$, we can choose one element, say x, from $f^{-1}(T)$. Then $f(x) = a$.
Therefore f is onto. Then f has a right inverse.

$\Rightarrow$: Assume that f has right inverse. We need to show that
Then we know f must be onto.
$T = f(f^{-1}(T))$ for all $T \subseteq A$.