Math 116 – Spring 2008, Text book: Elements of Modern Algebra by Gilbert & Gilbert.

SOLUTIONS TO SOME OF THE EXERCISES

1.4. 2. c) $*$ is a binary operation on $*$ defined by $x*y = x+2y$ for all $x, y \in \mathbb{Z}$.

Is $*$ commutative? No, $*$ is not commutative.

Because $1*2 = 1+2.2 = 5$ but $2*1 = 2+2.1 = 4$,

hence $1*2 \neq 2*1$, So, $*$ cannot be commutative.

Is $*$ associative? No, $*$ is not associative

because $(1*2)*3 = 5*3 = 5+6 = 11$

but $1*(2*3) = 1*(2+6) = 1+2.8 = 17 \neq 11$.

Hence $(1*2)*3 \neq 1*(2*3)$, so, $*$ is not associative.

Does $*$ have an identity element?

Suppose $e \in \mathbb{Z}$ is an identity element with respect to $*$.

Then $e*x = x$ for all $x \in \mathbb{Z}$ as e is an identity.

That is, $e+2x = x$ for all $x \in \mathbb{Z}$.

Then we obtain $e = x-2x = -x$ for all $x \in \mathbb{Z}$.

But, that implies $e = -1$ and $e = -2$, $e = -5$,

which in turn imply $-1 = -2 = -5$. This is a contradiction.

Therefore $*$ cannot have an identity element.

1.4. 8) Let A be a non-empty set, and $*$ be a binary operation on A. Suppose e is an identity element with respect to $*$.

If e' is also an identity element with respect to $*$,

then $e*e' = e'*e$. The last equality implies that

$e = e'$ because $e*e' = e$ when e' is regarded as an identity, $e*e' = e'$ when e is regarded as an identity.

Therefore, if there is an identity with respect to $*$, then that identity is unique.

1.4. 9). Assume that $*$ is an associative binary operation on A with an identity element e. Prove that the inverse of an element is unique when it exists.

Let $x \in A$, and $y \in A$ be an inverse for x.

Suppose $z \in A$ is also an inverse for x.

Then $y*x = x*y = e$ and $x*z = e$.

"Multiplying" both sides of the first equation by z

gives $(y*x)*z = (x*y)*z = e*z$.

Since $*$ is associative and e is an identity we obtain

$y*(x*z) = z$, The left hand side is $y*e = y$.

Therefore $y = z$, that is inverse is unique when it exists

1.4.10) (i) $f: \mathbb{Z} \to \mathbb{Z}$ is a function defined by $f(x) = |x|$ for $x \in \mathbb{Z}$

Does f have a right inverse? If yes, find one.

We know that a function f has a right inverse if and only if f is onto.

Let's check if f is onto:

~~Suppose f is onto.~~

Take $-5 \in \mathbb{Z}$. If f is onto, there should be $x_0 \in \mathbb{Z}$ such that $f(x_0) = |x_0| = -5$. Then $-5 \geq 0$. This is a contradiction because $-5 < 0$.

However, $|x| \geq 0$ for all $x \in \mathbb{Z}$.

This contradiction arises from the assumption that f is onto. Therefore f cannot be onto.

Hence, f can not have a right inverse.