A NATURAL OCCURENCE OF SHIFT EQUIVALENCE

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A natural occurence of shift equivalence in a purely algebraic setting constitutes the subject matter of the following short exposition.

Group endomorphisms $\alpha : G \longrightarrow G$, $\beta : H \longrightarrow H$, α, β are said to be *conjugate* if there exists an isomorphism $\theta : G \longrightarrow H$ such that $\theta \circ \alpha = \beta \circ \theta$. $\alpha : G \longrightarrow G$ and $\beta : H \longrightarrow H$ are said to be *shift equivalent* if there exist group endomorphisms $\varphi : G \longrightarrow H$, $\psi : H \longrightarrow G$ and $n \in \mathbf{Z}^+$ such that the relations

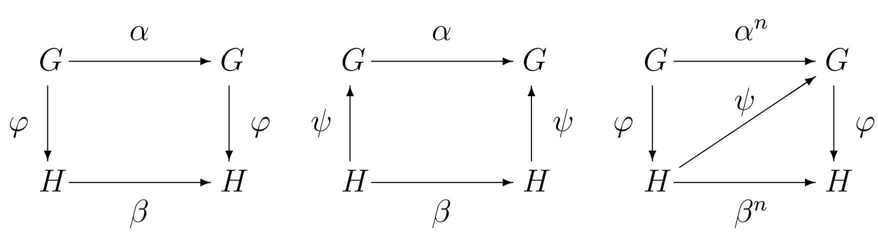
$$\varphi \circ \alpha = \beta \circ \varphi$$

$$\psi \circ \beta = \alpha \circ \psi$$

$$\psi \circ \varphi = \alpha^n$$

$$\varphi \circ \psi = \beta^n$$

hold, equivalently, the diagrams



commute. This state of affairs is described by saying that φ, ψ effect a shift equivalence of α to β of lag $n \in \mathbf{Z}^+$.

The concept of shift equivalence was introduced by R. F. Williams ([5], [6]) in the context of topological dynamics. The fact that shift equivalence is an equivalence relation among group endomorphisms can be demostrated by a straightforward argument. ([2])

Clearly both conjugacy and shift equivalence can be defined in any category and the former constitutes a special case of the former in two ways: (a) A shift equivalence with lag 0 is a conjugacy. (b) A shift equivalence between two automorphisms is a conjugacy.

The simple result presented here is due to myself and Ya. I. Ustinov ([1], [4]) independently. In my opinion this is the most straightforward and natural occurence of shift equivalence as complete invariant. Although by no means entirely novel, I feel that this elegant result deserves to be available to a wider public in the form of an independent exposition.

Another very natural occurence of shift equivalence arises in shape and homotopy theory. ([3])

Given a group endomorphism $\alpha : G \longrightarrow G$ the *simple direct limit* of α , denoted by $\mathcal{G} = \lim_{\rightarrow}(G, \alpha)$, is the set of equivalence classes in $G \times \mathbf{Z}^+$ under the equivalence relation $\sim$ where

$$(g, n) \sim (g', n')$$

iff

$$\alpha^{N-n}(g) = \alpha^{N-n'}(g')$$

for some $N \geq n, n'$.

$\sim$ can be easily checked to be an equivalence relation. $\mathcal{G}$ has a natural group structure with respect to the binary operation

$$(g, n)(g', n') = (\alpha^{n'}(g)\alpha^{n}(g'), n + n')$$

where, by abuse of notation, we let (g, n) stand for the equivalence class it represents. Again it can be routinely checked that this is a well-defined operator satisfying all group axioms. There are two natural isomorphisms on $\mathcal{G}$: Firstly,

$$\breve{\alpha} : \mathcal{G} \longrightarrow \mathcal{G}$$

defined by

$$\breve{\alpha}((g, n)) = (\alpha(g), n)$$

secondly,

$$s_\alpha : \mathcal{G} \longrightarrow \mathcal{G}$$

(which I like to call the "coshift") defined by

$$s_\alpha((g, n)) = (g, n + 1).$$

Again, well-definedness and morphology need checking. To see that $\breve{\alpha}$ and s_α are isomorphisms it is enough to observe that

$$\breve{\alpha} \circ s_\alpha = s_\alpha \circ \breve{\alpha} = Id(\mathcal{G})$$

PROPOSITION : ([1], [4])

Let G and H be finitely generated groups , $\alpha : G \longrightarrow G$, $\beta : H \longrightarrow H$ group endomorphisms, $\mathcal{G} = \lim_{\rightarrow}(G, \alpha)$, $\mathcal{H} = \lim_{\rightarrow}(H, \beta)$. The isomorphisms $s_\alpha : \mathcal{G} \longrightarrow \mathcal{G}$, $s_\beta : \mathcal{H} \longrightarrow \mathcal{H}$ are conjugate iff α, β are shift equivalent.

Proof : Given a subset K of agroup, let $\langle K \rangle$ denote the subgroup generated by K. There exist finite sets $A \subseteq G$, $B \subseteq H$ such that $G = \langle A \rangle$, $H = \langle B \rangle$. Assume first that s_α and s_β , or equivalently, $\check{\alpha}$ and $\check{\beta}$ are conjugate: There exists an isomorphism

$$T : \mathcal{G} \longrightarrow \mathcal{H}$$

such that

$$T \circ \check{\alpha} = \check{\beta} \circ T$$

Let

$$i_\alpha : G \longrightarrow \mathcal{G}$$

$$i_\beta : H \longrightarrow \mathcal{H}$$

be the natural injections defined by

$$i_\alpha(g) = (g, 0) \in \mathcal{G}$$

$$i_\beta(h) = (h, 0) \in \mathcal{H}$$

We have

$$T \circ i_\alpha(G) \subseteq \langle T \circ i_\alpha(A) \rangle$$

Clearly $T \circ i_\alpha(A)$ is a finite subset of $\mathcal{H}$. Hence there exists $k \in \mathbf{Z}^+$ such that

$$T \circ i_\alpha(G) \subseteq \langle T \circ i_\alpha(A) \rangle \subseteq H \times \{k\}$$

Therefore

$$\check{\beta}^k \circ T \circ i_\alpha(G) \subseteq H \times \{0\}$$

We define

$$\varphi = i_\beta^{-1} \circ \check{\beta}^k \circ T \circ i_\alpha : G \longrightarrow H$$

Similarly, there exists a sufficiently large $l \in \mathbf{Z}_+$ such that

$$\psi = i_\alpha^{-1} \circ \check{\alpha}^l \circ T \circ i_\alpha : H \longrightarrow G$$

is a well defined homomorphism. We claim that φ and ψ effect a shift equivalence of α to β with lag $k + l \in \mathbf{Z}^+$: Clearly

$$\varphi \circ \alpha = \beta \circ \varphi$$

$$\psi \circ \beta = \alpha \circ \psi$$

Moreover

$$\psi \circ \varphi = i_\alpha^{-1} \circ \check{\alpha}^l \circ T^{-1} \circ i_\beta \circ i_\beta^{-1} \circ \check{\beta}^k \circ T \circ i_\alpha = \alpha^{k+l}$$

Similarly

$$\varphi \circ \psi = \beta^{k+l}$$

Conversely assume, that there exist $\varphi : G \longrightarrow H$, $\psi : H \longrightarrow G$ and $n \in \mathbf{Z}^+$ such that $\varphi \circ \alpha = \beta \circ \phi$, $\psi \circ \beta = \alpha \circ \psi$, $\psi \circ \varphi = \alpha^n$, $\phi \circ \psi = \beta^n$. Consider the map

$$E : \mathcal{G} \longrightarrow \mathcal{H}$$

defined by

$$E((g, m)) = (\varphi(g), m)$$

and note that E is well-defined: If $\alpha^{l-m}(g) = \alpha^{l-m'}(g')$, then

$$\varphi \circ \alpha^{l-m}(g) = \varphi \circ \alpha^{l-m'}(g')$$

hence

$$\beta^{l-m} \circ \varphi(g) = \beta^{l-m'} \circ \varphi(g')$$

We have also $E \circ \check{\alpha} = \check{\beta} \circ E$ owing to $\varphi \circ \alpha = \beta \circ \varphi$, once again. Similarly define

$$F : \mathcal{H} \longrightarrow \mathcal{G}$$

by

$$F((h, m)) = (\psi(h), m)$$

We observe

$$\begin{aligned} F \circ E((g, m)) = F((\varphi(g), m)) &= (\psi \circ \varphi(g), m) \\ &= (\alpha^n(g), m) \\ &= \check{\alpha}^n(g, m) \end{aligned}$$

Thus $F \circ E = \check{\alpha}^n$. The right hand side is an isomorphism, E is an isomorphism, too, which commutes with $\check{\alpha}$ and $\check{\beta}$.

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