DIFFERENTIAL GEOMETRY

HOARD TWO

Cem Tezer

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Consider the torus $\mathbf{T}^2$ defined as the quotient space $\dfrac{\mathbb{R}^2}{\sim}$ where $(x, y) \sim (x', y')$ if $x' - x, y' - y \in \mathbb{Z}$. Write down the a map $h : \mathbf{T}^2 \longrightarrow \mathbb{R}^3$ which imbeds the torus in the usual way as the surface obtained by rotating the circle $(x - b)^2 + z^2 = a^2, y = 0$ about the z-axis, where $b > a > 0$.

Employ (x, y) as a chart to express the Riemannian metric that $\mathbf{T}^2$ inherits by means of this imbedding from the ordinary Riemannian manifold structure of $\mathbb{R}^3$.

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Let (x, y) be the identity chart on $\mathbb{R}^2$, (r, θ) be the chart "polar coordinates" . Compute the transition maps between the canonical charts $(x, y, \dot{x}, \dot{y})$, $(r, \theta, \dot{r}, \dot{\theta})$ on $T\mathbb{R}^2$.

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Let (x, y) be the identity chart on $\mathbb{R}^2$, (u, v) be the chart "parabolic coordinates". Compute the transition maps between the canonical charts $(x, y, \dot{x}, \dot{y})$, $(u, v, \dot{u}, \dot{v})$ on $T\mathbb{R}^2$.

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Let (x^1, x^2), (y^1, y^2), (z^1, z^2) be the usual "xy-", "yz-", "zx-charts" on $\mathbb{R}P^2$. Compute the transition maps between the canonical charts $(x^1, x^2, \dot{x}^1, \dot{x}^2)$, $(y^1, y^2, \dot{y}^1, \dot{y}^2)$, $(z^1, z^2, \dot{z}^1, \dot{z}^2)$ on $T\mathbb{R}P^2$.

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Let (x^1, x^2), (y^1, y^2), be the charts obtained by means of the stereographic projections from the south and north poles, let (θ, φ) be the chart "spherical polar coordinates" on $\mathbf{S}^2$. Compute the transition maps between the canonical charts $(x^1, x^2, \dot{x}^1, \dot{x}^2)$, $(y^1, y^2, \dot{y}^1, \dot{y}^2)$ and $(\theta, \varphi, \dot{\theta}, \dot{\varphi})$ on $T\mathbf{S}^2$.

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Prove that the function $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}$ defined by

$$f([x, y, z]) = \begin{cases} \exp\left(-\dfrac{x^2 + y^2}{z^2}\right) & \text{for} \quad z \neq 0 \\ \\ 0 & \text{for} \quad z = 0 \end{cases}$$

is a smooth scalar field. Use the standard charts on $\mathbb{R}P^2$ to express the covector field df.

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Use the standard charts to express the vector field on $\mathbb{R}P^2$, induced by the flow $\varphi : \mathbb{R}P^2 \times \mathbb{R} \longrightarrow \mathbb{R}P^2$ defined by

$$\varphi([x, y, z], t) = [x, e^t y, e^{-t} z].$$

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Consider the scalar field f on $\mathbb{R}P^2$ defined by

$$f([x,y,z]) = \frac{yz + 2zx + 5xy}{x^2 + y^2 + z^2}$$

(A) Express the covector field df in the standard charts.

(B) Let (x^1, x^2) be the standard chart declaring the line $x = 0$ to the line "at infinity" . Does there exist a covector field ω on $\mathbb{R}P^2$ such that

$$\omega|_{dom(x^1,x^2)} = \frac{x^1}{1 + (x^1)^2 + (x^2)^2} dx^1 + \frac{x^2}{1 + (x^1)^2 + (x^2)^2} dx^2 \quad ?$$

(C) Prove that there exists a(n obviously unique) covector field λ on $\mathbb{R}P^2$ such that

$$\lambda|_{dom(x^1,x^2)} = \frac{x^1}{(1 + (x^1)^2 + (x^2)^2)^2} dx^1 + \frac{x^2}{(1 + (x^1)^2 + (x^2)^2)^2} dx^2 \quad ?$$

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Let (θ, φ) be the chart "spherical polar coordinates" on $\mathbf{S}^2$.

(A) Prove that there exist smooth vector fields X, Y on $\mathbf{S}^2$ such that

$$X|_{dom(\theta,\varphi)} = \frac{\partial}{\partial \theta} \quad , \quad Y|_{dom(\theta,\varphi)} = \sin\varphi \, \frac{\partial}{\partial \varphi}$$

(B) Prove that there exist smooth covector fields ξ, η on $\mathbf{S}^2$ such that

$$\xi|_{dom(\theta,\varphi)} = \sin^2\varphi \; d\theta \quad , \quad \eta|_{dom(\theta,\varphi)} = \sin\varphi \cos\varphi \; d\varphi$$

(C) Use the charts obtained by stereographic projection to express the smooth functions $\xi(X), \eta(X), \xi(Y), \eta(Y)$ on $\mathbf{S}^2$.

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Consider the flow $\varphi : \mathbf{S}^2 \times \mathbb{R} \longrightarrow \mathbf{S}^2$ defined by

$$\varphi((x,y,z),t) = \left(\frac{2xe^t}{1+z+(1-z)e^{2t}} \; , \; \frac{2ye^t}{1+z+(1-z)e^{2t}} \; , \; \frac{1+z-(1-z)e^{2t}}{1+z+(1-z)e^{2t}} \right)$$

Let X be the vector field on $\mathbf{S}^2$ generated by φ.

(A) Express X in the chart (x^1, x^2) obtained by stereographic projection from the south pole on $\mathbf{S}^2$.

(B) Express X in the chart (θ, φ) the "spherical polar coordinates" .

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Express the following elements of $T^{(1,p)}\mathbb{R}^n$ as multilinear maps from $\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ to $\mathbb{R}^n$ by describing their effect on the standard bases :

(A) $e_2 \otimes e^1 - 2e_1 \otimes e^1 \in T^{(1,1)}(\mathbb{R}^2)$.

(B) $3e_1 \otimes e^2 \otimes e^2 - 5e_2 \otimes e^2 \otimes e^1 \in T^{(1,2)}(\mathbb{R}^2)$.

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Introduce the parabolic coordinates (u, v) on $\mathbb{R}^2$ with

$$(u,v) : dom(u,v) = \mathbb{R}^2 - \mathbb{R}_{.\geq 0} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}_{.>0}$$

and

$$x(u,v) = u^2 - v^2 \quad , \quad y = 2uv \ .$$

Express

$$\frac{\partial}{\partial x} \otimes dx + 3\frac{\partial}{\partial y} \otimes dx - 2\frac{\partial}{\partial x} \otimes dy \in T^{(1,1)}(T_{(-1,0)}\mathbb{R}^2)$$

in (u, v).

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Consider the following elements of $\mathcal{X}^{(2,0)}(\mathbb{R}^2)$ and decide if they are Riemannian metrics, semi-riemannian metrics or none ?

(A) $(1 + x^2)dx \otimes dx + dy \otimes dy$.

(B) $x^2 dx \otimes dy - y^2 dy \otimes dx$.

(C) $(1 + x^2)dx \otimes dx + (1 + y^3)dy \otimes dy$.

(D) $(1 + x^2)dx \otimes dx - dy \otimes dy$.

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Let (z^1, z^2) be the standard chart on $\mathbb{R}P^2$ which declares $z = 0$ to the line at infinity.

(A) Prove that there exists a(n obviously unique) tensor field $\mathsf{G} \in \mathcal{X}^{(2,0)}(\mathbb{R}P^2)$ such that

$$\mathsf{G}|_{dom(z^1,z^2)} = \frac{(1+(z^2)^2)dz^1 \otimes dz^1 - 2z^1z^2 dz^1 \otimes dz^2 + (1+(z^1)^2)dz^2 \otimes dz^2}{[1+(z^1)^2+(z^2)^2]^2}.$$

(B) Prove that G is a Riemannian tensor.

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Let (u^1, u^2) be the chart on $\mathbf{S}^2$ obtained by means of the stereographic projection from the south pole.

(A) Prove that there exists a Riemannian metric on $\mathbf{S}^2$ with

$$\mathsf{G}|_{dom(u^1,u^2)} = \frac{du^1 \otimes du^1 \ + \ du^2 \otimes du^2}{[1+(u^1)^2+(u^2)^2]^2} \quad .$$

(B) Prove that $\mathbf{S}^2$ admits a tensor field Θ of bidegree $(1,1)$ with

$$\Theta|_{dom(u^1,u^2)} = \frac{\partial}{\partial u^2} \otimes du^1 - \frac{\partial}{\partial u^1} \otimes du^2$$

(C) Prove that

$$\mathsf{G}(\Theta(X), X) = 0$$

$$\Theta^2(X) = -X$$

$$\mathsf{G}(\Theta(X), \Theta(Y)) = \mathsf{G}(X, Y)$$

for all $X, Y \in \mathcal{X}(\mathbf{S}^2)$.

(D) Give a simple geometric interpretation of Θ.

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Employing charts of your own choice write down a tensor field Θ of bidegree $(1,1)$ on the ordinary sphere $\mathbf{S}^2$ with its usual Riemannian metric G such that

$$\Theta(X) \neq X$$

$$\Theta^3(X) = X$$

$$\mathsf{G}(\Theta(X), \Theta(Y)) = \mathsf{G}(X, Y)$$

for any vector fields X, Y on $\mathbf{S}^2$.