GENERAL TOPOLOGY

HEAP TWO

Cem Tezer

$\boxed{\mathbf{1}}$

Let X be a topological space. Given $A \subseteq X$, let $\overline{A}$ denote the closure of A.

(A) For a finite family $\mathcal{A}$ of subsets of X, prove that

$$\overline{\cup\mathcal{A}} = \cup\{\overline{A} \mid A \in \mathcal{A}\} \quad .$$

(B) Give a counterexample to show that the above identity does not hold for an arbitrary family $\mathcal{B}$ of subsets of X, in general.

(C) A family $\mathcal{M}$ of subsets is said to be *locally finite* if each $x \in X$ has a neighbourhood V such that the set

$$\{M \in \mathcal{M} \mid M \cap V \neq \emptyset\}$$

has finite cardinality. Prove that the identity in (A) holds for a locally finite family.

2

Let X be a topological space. Given $A \subseteq X$, let $\overline{A}$ denote the closure of A. If $C \subseteq X$ is dense and $U \subseteq X$ is open, prove that

$$U \subseteq \overline{(U \cap C)}$$

3

(A) Prove that the intersection of all T_1 topologies on a set X is exactly the cofinite topology on X.

(B) Given topological spaces X, Y, learn about the topology on $X \times Y$. Prove that a topological space Y is Hausdorff iff $diag(Y) = \{(y, y) \mid y \in Y\}$ is closed in $Y \times Y$.

(C) Let Z be a finite set. Prove that there is only one Hausdorff topology on Z.

4

Let X be an ordered set. To be precise, there is a relation $\geq$ on X which is reflexive, anti-symmetric, transitive. The *right order topology* on X is the topology which admits as a base the family $\mathcal{R} = \{R_x \ : \ x \in X\}$ where for each $x \in X$

$$R_x = \{y \in X : y \geq x\}$$

(A) Prove that the right order topology is a T_0 topology in which the intersection of an arbitrary family of open sets is open.

(B) Prove that $y \geq x$ iff $x \in \overline{\{y\}}$ with respect to the right order topology on X.

(C) Conversely, given any T_0 topology $\mathcal{J}$ on X in which the intersection of an arbitrary family of open sets is open, prove that the relation ρ on X where $y \rho x$ is defined by $x \in \overline{\{y\}}$, is an order relation with respect to which $\mathcal{J}$ is exactly the right order topology.

5

(A) Prove that the product of T_1 spaces is a T_1 space .

(B) Let X, Y be topological spaces, $q : X \longrightarrow Y$ be a quotient map. If X is T_1, prove that Y is a T_1 space iff $q^{-1}(\{y\})$ is closed, for every $y \in Y$.

(C) Give an example of a quotient map $q : X \longrightarrow Y$ such that X is T_1 whereas Y is not T_1.

6

Given a metric space (X, d) and $a \in X$, $r > 0$, let

$$V(a,r) = \{x \in X \mid d(a,x) < r\}$$

$$V[a,r] = \{x \in X \mid d(a,x) \leq r\} \quad .$$

For $M \subseteq X$, let $\overline{M}$ denote the closure of M in X.

(A) Prove that $\overline{V(a,r)} \subseteq V[a,r]$ for any $a \in X$ and $r > 0$.

(B) For each $a \in X$, prove that the function $f : X \longrightarrow \mathbb{R}$ defined by $f(x) = d(x,a)$ for $x \in X$, is continuous.

(C) Suppose that X is the union of the <u>disjoint</u> sets $A, K \subseteq X$ where A is closed and K is compact. Prove that for each $a \in A$ there exists $r > 0$ such that

$$\overline{V(a,r)} \neq V[a,r]$$

7

Given a metric space (X, d) and $a \in X$, $r > 0$, let

$$V(a, r) = \{x \in X \mid d(a, x) < r\}$$

$$V[a, r] = \{x \in X \mid d(a, x) \leq r\} \quad .$$

For $M \subseteq X$, let $\overline{M}$ denote the closure of M in X.

(A) Prove that $\overline{V(a, r)} \subseteq V[a, r]$ for any $a \in X$ and $r > 0$.

(B) For each $a \in X$, prove that the function $f : X \longrightarrow \mathbb{R}$ defined by $f(x) = d(x, a)$ for $x \in X$, is continuous.

(C) Suppose that X is the union of the disjoint sets $A, K \subseteq X$ where A is closed and K is compact. Prove that for each $a \in A$ there exists $r > 0$ such that

$$\overline{V(a, r)} \neq V[a, r]$$

Let X be a Hausdorff topological space. Remember that given $A \subseteq X$, a point $p \in X$ is said to be an *accumulation point* of A if $(V - \{p\}) \cap A \neq \emptyset$ for every neighbourhood V of p.

(A) Prove that an infinite compact $A \subseteq X$ contains at least one accumulation point of itself.

(B) Suppose that there exists a (not necessarily continuous !) map

$$f : X \longrightarrow \mathbb{R} - \{0\}$$

such that

$$\lim_{x \longrightarrow p} \frac{1}{f(x)} = 0$$

for each accumulation point p of X. Prove that $K \cap f^{-1}([a, b])$ is a finite set for every compact $K \subseteq X$ and $a, b \in \mathbb{R}$.

(C) Prove that every infinite compact subset of X is countable. Given a topological space X, a set $A \subseteq X$ is said to be a *retract* of X if there exists a continuous map $r : X \longrightarrow A$ such that $r|_A = Id_A$.

(a) Prove that a retract of a normal space is normal.

(b) Let X, Y be topological spaces and $f : X \longrightarrow Y$ be a continuous map. Prove that the graph of f, that is, the set $graph(f) = \{(x, f(x)) \mid x \in X\}$ is a retract of $X \times Y$.

(c) Is a retract necessarily closed ?

Genaral Topology

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MIDTERM

23rd November 2001

[5 + 5 + 10] , [5 + 5 + 10] , [5 + 5 + 10 + 10] , [10 + 10 + 10]

1.

(A) Let X be an infinite set with the cofinite topology. Prove that $A \subseteq X$ is dense iff it is infinite.

(B) Given an infinite set Y, prove that the cofinite topology is the largest topology on Y with respect to which all infinite subsets of Y are dense.

(C) Given an infinite set Z, what is the topology on Z with respect to which the only dense subset of Z is Z itself ?

2.

A topological space X is said to be *symmetric* if for any $x, y \in X$, $x \in \overline{\{y\}}$ holds iff $y \in \overline{\{x\}}$ holds.

(A) Give an example of a space which is symmetric but not Hausdorff.

(B) Give an example of a space which is normal but not regular.

(C) Prove that a symmetric normal space is regular.

3.

(A) Let X, Y be topological spaces, Y be Hausdorff, $f, g : X \longrightarrow Y$ be continuous functions. Prove that

$$\{x \in X \mid f(x) = g(x)\}$$

is a closed set.

(B) Let Z be a Hausdorff space, $h : Z \longrightarrow Z$ be continuous functions. Prove that

$$Fix(h) = \{x \in X \mid h(x) = x\}$$

is a closed set.

(C) Given an open interval $\mathrm{I} = (a, b) \subseteq \mathbb{R}$, write down a homeomorphism $f : (a, b) \longrightarrow (a, b)$ such that $Fix(f) = \emptyset$ and

$$\lim_{x \longrightarrow a^+} f(x) = a \ , \quad \lim_{x \longrightarrow b^-} f(x) = b$$

(D) Given a closed $K \subseteq (0, 1)$, prove that there exists a homeomorphism $f : [0, 1] \longrightarrow [0, 1]$ such that $Fix(f) = K$.

4.

(A) Let X, Y be topological spaces. Given compact $K \subseteq X$, $y \in Y$ and an open $W \subseteq X \times Y$ such that $K \times \{y\} \subseteq W$, prove that y has a neighbourhood $V \subseteq Y$ such that $K \times V \subseteq A$.

(B) If X is compact, prove that the projection $p : X \times Y \longrightarrow Y$ onto the second component is a closed map.

(C) Prove that a topological space Z is compact iff the projection $p : Z \times M \longrightarrow M$ onto the second component is a closed map for every topological space M.

Genaral Topology

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FINAL EXAMINATION

10th January 2002

[6 + 8 + 9 + 5] , [10 + 6] , [6 + 8 + 7 + 7] , [6 + 6 + 8 + 8]

1.

A topological space is called a *door space* if every subset thereof is either open or closed or both.

(A) In a topological space X, prove that a point $a \in X$ fails to be an accumulation point iff the set $\{a\}$ is open.

(B) Let Y be a door space. If the point $b \in Y$ is an accumulation point, prove that for any neighbourhood N of b the set $N - \{b\}$ is open.

(C) Prove that a Hausdorff door space has at most one accumulation point.

(D) Give examples of (1) a Hausdorff door space with no accumulation point, (2) a Hausdorff door space with exactly one accumulation point, (3) a door space with at least two accumulation points.

2.

(A) Prove that in a regular space the closure of a compact set is compact.

(B) Give examples of (1) a topological space which has a proper compact subset whereof the closure is not compact (2) a topological space which is not regular but in which the closure of each compact set is compact.

3.

A subset of a topological space is referred to as a *G_δ-set* if it is the intersection of a countable family of open sets. A topological space is said to be *perfectly normal* if it is normal and every closed subset thereof is a G_δ-set.

(A) Given a topological space X and a continuous map $f : X \longrightarrow \mathbb{R}$, prove that $f^{-1}(t)$ is a G_δ-set for every $t \in \mathbb{R}$.

(B) If Y is a normal space, prove that for every closed G_δ-set $B \subseteq Y$, there exists a continuous function $g : Y \longrightarrow \mathbb{R}$ such that $B = g^{-1}(0)$.

(C) Prove that a pseudometrisable space is perfectly normal.

(D) Is $[0, 1]^{\mathbb{R}}$ perfectly normal ? (*Hint* : Consider singletons !)

4.

Given a metric space (X, d), $r > 0$ and any $A \subseteq X$, let $V_r(A)$ be defined by

$$V_r(A) = \bigcup_{a \in A} V_r(a)$$

where, as usual

$$V_r(x) = \{y \in X \mid d(x, y) < r\}$$

for any $x \in X$.

Suppose that the metric space (X, d) is bounded in the sense that $X = V_R(x_0)$ for some $x_0 \in X$ and $R > 0$. Let $\tilde{X}$ stand for the set of closed subsets of X. Consider the function $\tilde{d} : \tilde{X} \times \tilde{X} \longrightarrow \mathbb{R}$ defined by

$$\tilde{d}(K, L) = \inf\{r > 0 \mid K \subseteq V_r(L), L \subseteq V_r(K)\}.$$

for $K, L \in \tilde{X}$.

(A) Prove that $\tilde{d}$ is a metric on $\tilde{X}$.

(B) Let $Y = \mathbb{R}_{>0} = \{t \in \mathbb{R} \mid t > 0\}$. Consider d_1, $d_2 : Y \times Y \longrightarrow \mathbb{R}$ defined by

$$d_1(s, t) = |\frac{s}{1+s} - \frac{t}{1+t}| \ ,$$

$$d_2(s, t) = \min\{1, |s - t|\}$$

for $s, t \in \mathbb{R}$. Prove that d_1, d_2 are metrics which induce the usual topology on Y.

(C) Consider $\tilde{d}_1, \tilde{d}_2$ on $\tilde{Y}$. Let $\tilde{y}_n = \{1, 2, 3, \ldots, n\}$ for each $n \in \mathbb{N}$. Prove that $\lim_{n \longrightarrow \infty} \tilde{y}_n = \mathbb{N}$ with respect to $\tilde{d}_1$.

(D) Do $\tilde{d}_1$ and $\tilde{d}_2$ induce the same topology on $\tilde{Y}$?