HOMEWORK

I. Let U be a vector space. $\Omega \in \bigwedge^p(\mathrm{U}^*)$ is said to be *divisible* by $\lambda \in \bigwedge^q(\mathrm{U}^*)$ if there exists $\mu \in \bigwedge^{p-q}(\mathrm{U}^*)$ such that $\Omega = \lambda \wedge \mu$.

(A) Prove that $\Omega \in \bigwedge^q(\mathrm{U}^*)$ is divisible by $\lambda \in \bigwedge^1(\mathrm{U}^*)$ iff $\Omega \wedge \lambda = 0$.

(B) Let $\lambda, \mu \in \bigwedge^1(\mathrm{U}^*)$ be linearly independent. Prove that Ω is divisible by $\lambda, \mu \in \bigwedge^1(\mathrm{U}^*)$ iff Ω is divisible by $\lambda \wedge \mu \in \bigwedge^2(\mathrm{U}^*)$.

II. Let U be a 4-dimensional vector space, $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_4\}$ be an arbitrary basis of $\mathbb{R}^4$. Consider

$$\varphi = A\mathsf{e}_1 \wedge \mathsf{e}_2 + B\mathsf{e}_1 \wedge \mathsf{e}_3 + C\mathsf{e}_1 \wedge \mathsf{e}_4 + D\mathsf{e}_2 \wedge \mathsf{e}_3 + E\mathsf{e}_2 \wedge \mathsf{e}_4 + F\mathsf{e}_3 \wedge \mathsf{e}_4 \in \bigwedge^2(\mathrm{U}) \quad .$$

(A) If $AF - BE + CD \neq 0$, prove that every $\Theta \in \bigwedge^3(\mathrm{U})$ is divisible by φ.

(B) If $AF - BE + CD = 0$, prove that the non-zero

$$\Theta = a\ \mathsf{e}_2 \wedge \mathsf{e}_3 \wedge \mathsf{e}_4 \ +\ b\ \mathsf{e}_3 \wedge \mathsf{e}_4 \wedge \mathsf{e}_1 \ +\ c\ \mathsf{e}_4 \wedge \mathsf{e}_1 \wedge \mathsf{e}_2 \ +\ d\ \mathsf{e}_1 \wedge \mathsf{e}_2 \wedge \mathsf{e}_3 \in \bigwedge^3(\mathrm{U})$$

is divisible by φ iff the matrix

$$\begin{bmatrix} 0 & F & -E & D & a \\ -F & 0 & C & -B & -b \\ E & -C & 0 & A & c \\ -D & B & -A & 0 & -d \end{bmatrix}$$

has rank 2.

III. Let U be a vector space. Prove that $\omega \in \bigwedge^2(\mathrm{U}^*)$ is non-degenerate iff $dim\mathrm{U} = 2n$ for some n and

$$\omega^{\wedge n} = \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}} \neq 0$$

IV. Let H be a tensor field of bidegree $(0, 2)$ on the manifold M. For every fixed $X \in \mathfrak{X}(M)$, prove that $\mathsf{K} = \mathsf{K}_X \ : \ \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{F}(M)$ defined by

$$\mathsf{K}(Y, Z) = X\mathsf{H}(Y, Z) - \mathsf{H}([X, Y], Z) - \mathsf{H}(Y, [X, Z])$$

for $Y, Z \in \mathfrak{X}(M)$ is a tensor field of bidegree $(0, 2)$ on M .

V. Given a tensor field S of bidegree $(1, 1)$ on M with the property $\mathsf{S} \circ \mathsf{S} = -Id$, prove that $\mathsf{N} \ : \ \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ defined by

$$\mathsf{N}(X, Y) = [\mathsf{S}(X), \mathsf{S}(Y)] - \mathsf{S}([\mathsf{S}(X), Y]) - \mathsf{S}([X, \mathsf{S}(Y)]) - [X, Y]$$

for $X, Y \in \mathfrak{X}(M)$ is a tensor field of bidegree $(1, 2)$ on M .