HOMEWORK ONE

I. Let H be a tensor field of bidegree $(0,2)$ on the manifold M.

(A) Prove that

$$\mathrm{L}_X\mathsf{H} = X\mathsf{H}(Y,Z) - \mathsf{H}([X,Y],Z) - \mathsf{H}(Y,[X,Z])$$

(B) Let $x = (x^i)_{1\le i\le n}$ be a chart on M. If

$$A|_{dom(x)} = A^\alpha \frac{\partial}{\partial x^\alpha}$$

and

$$\mathsf{H}|_{dom(x)} = \mathsf{H}_{ij} dx^i \otimes dx^j \quad ,$$

prove that

$$\mathrm{L}_A\mathsf{H}|_{dom(x)} = \left[A^\alpha \frac{\mathsf{H}_{ij}}{\partial x^\alpha} + \frac{\partial A^\alpha}{\partial x^i}\mathsf{H}_{\alpha j} + \frac{\partial A^\alpha}{\partial x^j}\mathsf{H}_{i\alpha} \right] dx^i \otimes dx^j \quad ,$$

(C) Given a Riemannian manifold (M,G), a vector field $X \in \mathcal{X}(M)$ is called a *Killing vector field* if $\mathrm{L}_X\mathsf{G} = 0$. Prove that $A \in \mathcal{X}(\mathbb{R}^2)$ is a Killing vector field on $(\mathbb{R}^2, dx\otimes dx + dy\otimes dy)$ iff

$$A = (-\omega y + a)\frac{\partial}{\partial x} + (\omega x + b)\frac{\partial}{\partial y}$$

for some $\omega, a, b \in \mathbb{R}$.

(D) Consider the Riemannian tensor field

$$\tilde{\mathsf{G}} = \alpha\ dx\otimes dx + F(x)\ \left(\ \beta\ dy\otimes dy + \gamma\ dz\otimes dz\ \right)$$

on $\mathbb{R}^3$ where α, β, γ are constants. Prove that the vector space of Killing vector fields of $\tilde{\mathsf{G}}$ is generated by the vector fields

$$\frac{\partial}{\partial y}\ ,\ \frac{\partial}{\partial z}\ \text{ and }\ \gamma\, z\frac{\partial}{\partial y} - \beta\, y\frac{\partial}{\partial z} \quad .$$