MATH 373 - GEOMETRIES I

FIRST MIDTERM

FAMILY NAME	OTHER NAMES	GRADE

14th October 2005. Duration : 110 minutes. Three questions : 10 + 15 + 10 , 5 + 25 , 10 + 15+ 10

Solutions

1. Prove that in a triangle ABC the following conditions are equivalent :

$$\text{(i)}\quad \alpha = \frac{1+\cos A}{1-\cos A}$$

$$\text{(ii)}\quad \alpha r + r_b + r_c = \alpha r_a$$

$$\text{(iii)}\quad r_b r_c = \alpha r r_a$$

Remember that Δ (area of ABC) $= rs = r_a(s-a) = r_b(s-b) = r_c(s-c)$.

(ii) iff (iii) :

$$r_b + r_c = \alpha(r_a - r) \text{ iff } \frac{1}{s-b} + \frac{1}{s-c} = \alpha\left(\frac{1}{s-a} - \frac{1}{s}\right)$$

$$\text{iff } \frac{2s-b-c}{(s-b)(s-c)} = \alpha \frac{a}{s(s-a)} \quad (2s-b-c = a)$$

$$\text{iff } \frac{1}{(s-b)}\frac{1}{(s-c)} = \alpha \frac{1}{s(s-a)} \text{ iff } r_b r_c = \alpha r r_a.$$

(iii) iff (i) :

$$\alpha(s-b)(s-c) = s(s-a)$$

iff

$$\alpha(a-(b-c))(a+(b-c)) = (a+(b+c))((b+c)-a)$$

$$\alpha(a^2-(b-c)^2) = (b+c)^2 - a^2$$

$$\alpha(2bc - (b^2+c^2-a^2)) = \underbrace{b^2+c^2-a^2}_{2bc\cos A} + 2bc$$

$$(b^2+c^2-a^2 = 2bc\cos A)$$

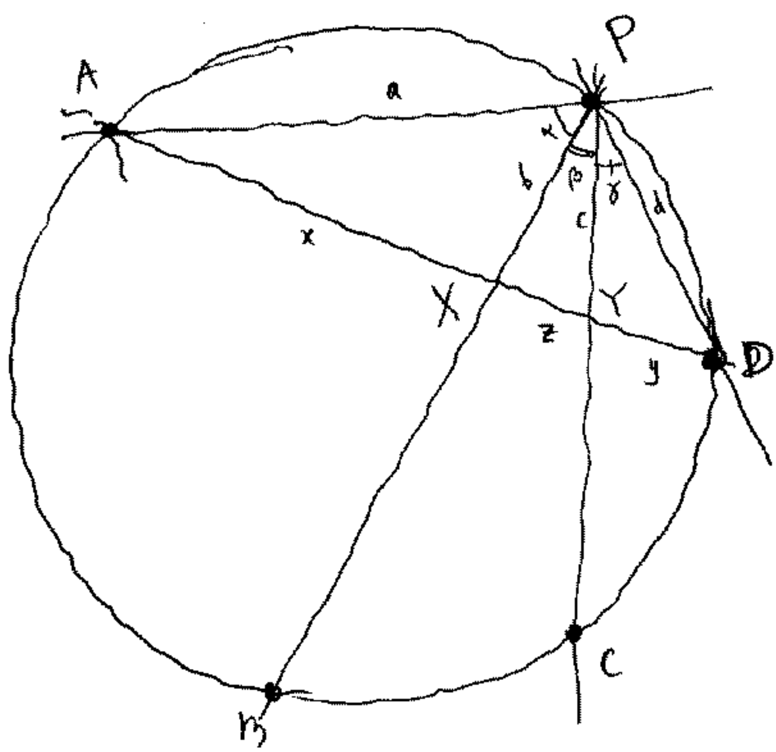
$$\text{iff } \alpha = \frac{1+\cos A}{1-\cos A}.$$

2. (A) In a positively oriented triangle ABC with area Δ, prove that

$$\Delta = \frac{1}{2}\, bc \sin A$$

(B) Given a circle Γ, be distinct fixed points $A, B, C, D \in \Gamma$ and a variable point $P \in \Gamma$, let PB, PC intersect AD in X, Y respectively and put $x = |AX|$, $y = |DY|$, $z = |XY|$. By computing and comparing the areas of the triangles PAX, PYD, PXY, and PAD or otherwise prove that xyz^{-1} is independent of the position of $P \in \Gamma$.

(A) Routine.



$$\underbrace{\frac{xy}{z\underbrace{(x+y+z)}_{|AD|}}}\ = \frac{\text{Area}(PAX)\ \text{Area}(PYD)}{\text{Area}(PXY)\ \text{Area}(PAD)} = \frac{\frac{1}{2}ab\sin\alpha\ \frac{1}{2}cd\sin\gamma}{\frac{1}{2}bc\sin\beta\ \frac{1}{2}ad\sin(\alpha\cdot}$$

$|AD|$ is constand, α, β, γ are constant.

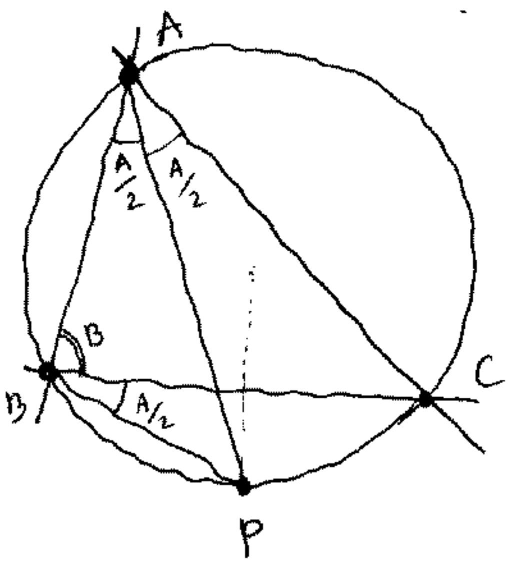
$\therefore \quad \dfrac{xy}{z}$ is constant...

3. Let P be a point on the circumcircle of a triangle ABC such that AP is the internal bisector of the angle A. Prove that the following conditions are equivalent :

(i) $|PA| = 2|PB|$

(ii) $2\sin\left(\frac{A}{2}\right) = \cos\left(\frac{B-C}{2}\right)$

(iii) $2a = b + c$



(i) iff (ii) :

By the sine rule in ABP :

$|PB| = 2R\sin\frac{A}{2}$

$|PA| = 2R\sin\left(\frac{A}{2} + B\right)$

$= 2R\cos\left(\frac{B-C}{2}\right)$

Thus $|PA| = 2|PB|$ iff

$2\sin\frac{A}{2} = \cos\left(\frac{B-C}{2}\right)$

(ii) iff (iii) : $\sin\frac{A}{2} = \cos\left(\frac{B-C}{2}\right)$ iff

$2\sin A = \cos\left(\frac{B-C}{2}\right)\cos\frac{A}{2}$

$= \sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$

$= \sin\left(\frac{B+C}{2} + \frac{B-C}{2}\right) + \sin\left(\frac{B+C}{2} - \frac{B-C}{2}\right)$

$= \sin B + \sin C$

iff $2a = b + c$ by the sine rule in ABC.