Geometry I

MATH 373

SECOND MIDTERM

(*Duration : 110 mins.*)

17th December 2004

[20] , [25 + 15] , [10 + 10] , [20]

Hurşit Önsiper tarafından, sorularin eksik basılması münasebetiyle !

İlm bir lücce-i bî-sahildir
Anda alim geçinen cahildir !
Nabî

ve kesb-i ilm, talim-i ilm içün itina gerekdir.

1.

Write down the equation of the parabola of focus $F(1,1)$ and directrix $x+y=0$.

$$2\left(\frac{x+y}{2}\right)^2 = (x-1)^2+(y-1)^2$$

$$\boxed{(x-y)^2-4(x+y-1)=0}$$

2.

(A) Write down the equation of the conic section with focus $F(c, 0)$, directrix $x = d$ and eccentricity $\varepsilon > 0$

(B) Choosing d, c, ε suitably , show that $x^2 - y^2 = 1$ represents a hyperbola.

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(A) $$\frac{\left((x-c)^2 + y^2\right)^{1/2}}{|x-d|} = \varepsilon$$

which gives

$$\boxed{(\varepsilon^2 - 1)x^2 - y^2 + 2(c - \varepsilon^2 d)x + \varepsilon^2 d^2 - c^2 = 0}$$

(B) $\varepsilon = \sqrt{2} > 1$, $d = 1/\sqrt{2}$ $c = \sqrt{2}$

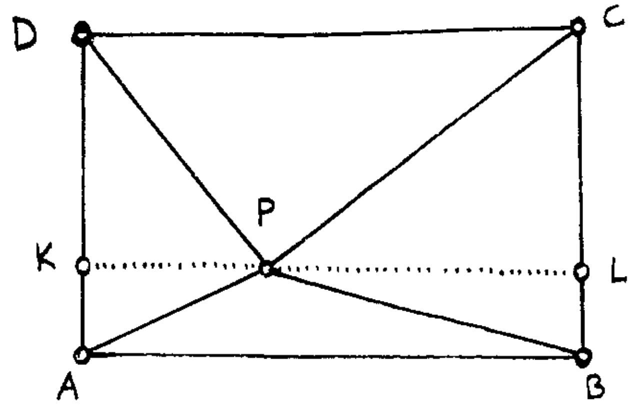
(A) Given a rectangle $ABCD$ prove that

$$|PA|^2 + |PC|^2 = |PB|^2 + |PD|^2 \quad .$$

for any point P.

(B) Let φ be an ellipse. Prove that the set of points from which the tangents to φ are perpendicular to one another, is a circle.

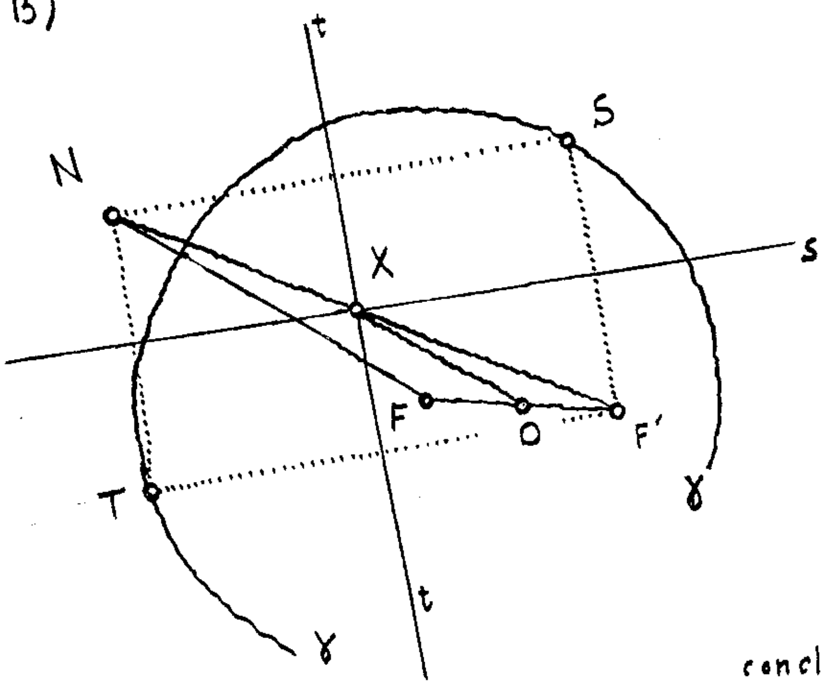
A)



Let K, L be the feet of the perpendiculars from P onto AD, BC respectively:

$$|PA|^2 - |PD|^2 = |AK|^2 - |KD|^2 = |BL|^2 - |LC|^2 = |PB|^2 - |PC|^2$$

B)

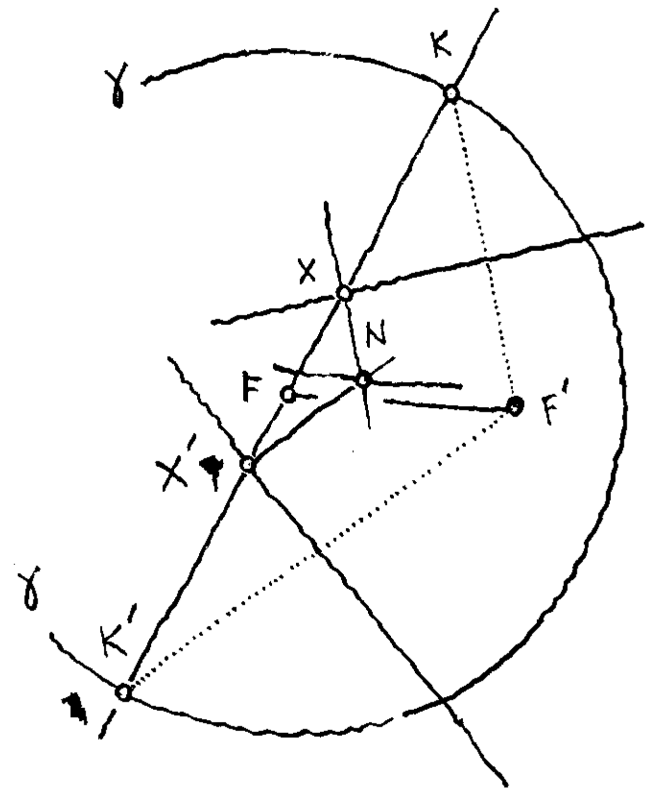


Let F, F' be the foci of φ, O be the midpoint of $[F, F']$. Consider tangents s, t to φ which intersect in X and which are perpendicular to one another. The respective reflections ~~&~~ S, T of F' in s, t lie on the directrix circle γ with center F. Consider the rectangle $F'SNT$ to conclude $|FN|^2 + |FF'|^2 = 2|FS|^2$

hence $|FN| = \text{constant}$. Therefore $|OX| = \frac{1}{2}|FN| = \text{constant}$.

4.

Consider an ellipse φ with foci F, F'. Let a line through F meet φ in X, X'. If the normals to φ at X, X' intersect in N, prove that the parallel to FF' through N bisects $[X, X']$.



Let K, K' be the respective reflections of F' in the tangents to φ through X, X'. Clearly $XN /\!/ KF'$, $X'N /\!/ K'F'$. K, K' lie on the directrix circle γ with centre F. ~~Er~~ FF' bisects $[K, K']$. Consequently the parallel through N to FF' bisects $[X, X']$.