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Geometry I

M A T H 3 7 3

FIRST MIDTERM

(*Duration : 110 mins.*)

18th November 2004

[15 + 15] , [15 + 20] , [10 + 10] , [15]

Ayinesi iştir kişinin lafa bakılmaz
~~Görülür kişinin~~ rütbe-i aklı eserinde !
Şahsın görünür

Ziya Paşa

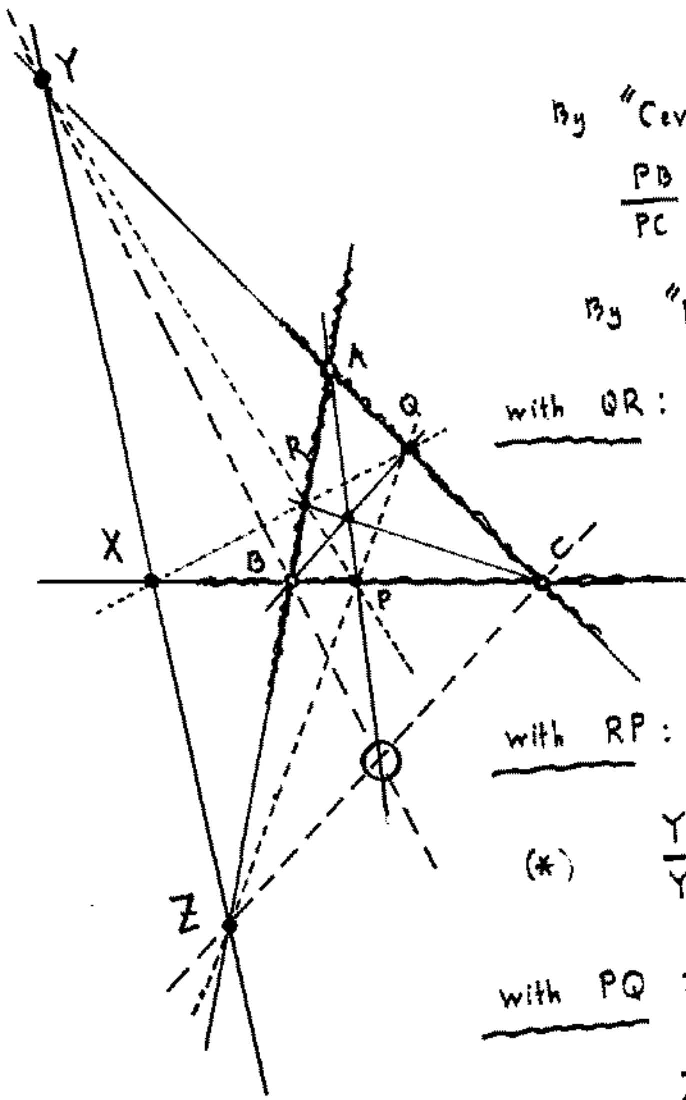
1.

Given a triangle ABC, consider $P \in BC - \{B, C\}$, $Q \in CA - \{C, A\}$, $R \in AB - \{A, B\}$ such that AP, BQ, CR are concurrent. Let QR, RP, PQ meet BC, CA, AB in X, Y, Z respectively. Prove that

(A) X, Y, Z are collinear.

(B) AP, BY, CZ are concurrent, or parallel.

P L E A S E T U R N O V E R !



By "Ceva" in ABC :

$$\frac{PB}{PC}\cdot\frac{QC}{QA}\cdot\frac{RA}{RB} = -1$$

By "Menelaus" in ABC

with QR : $$\frac{XB}{XC}\,\frac{QC}{QA}\,\frac{RA}{RB} = +1$$

with RP :

(*) $$\frac{YC}{YA}\,\frac{RA}{RB}\,\frac{PB}{PC} = +1$$

with PQ :

(**) $$\frac{ZA}{ZB}\,\frac{PB}{PC}\,\frac{QC}{QA} = +1$$

Multiplying ~~all~~ the last three up, we obtain

$$\frac{XB}{XC}\,\frac{YC}{YA}\,\frac{ZA}{ZB} = \left(\frac{PB}{PC}\,\frac{QC}{QA}\,\frac{RA}{RB}\right)^{-2} = (-1)^{-2} = +1 .$$

We conclude by "Menelaus" in ABC, that X, Y, Z are collinear.

Multiplying (*) and (**) we obtain

$$\frac{PB}{PC}\,\frac{YC}{YA}\,\frac{ZA}{ZB} = \left(\frac{PB}{PC}\,\frac{QC}{QA}\,\frac{RA}{RB}\right)^{-1} = (-1)^{-1} = -1 .$$

We conclude by "Ceva" in ABC, that AP, BY, CZ are concurrent ~~and~~ or parallel.

2.

Consider the triangle ABC with the usual notation.

(A) Prove that

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r} .$$

(B) Prove that

$$\frac{1}{r^2}+\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}=4\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right) .$$

(A) $$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{\Delta}\left((s-a)+(s-b)+(s-c)\right)=\frac{1}{\Delta}\left(3s-\underbrace{(a+b+c)}_{2s}\right)=\frac{s}{\Delta}=\frac{1}{r}$$

$$=\frac{s}{\Delta}=\frac{a+b+c}{2\Delta}=\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}$$

(B) $$\frac{1}{r^2}+\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}=\frac{1}{\Delta^2}\left(s^2+(s-a)^2+(s-b)^2+(s-c)^2\right)$$

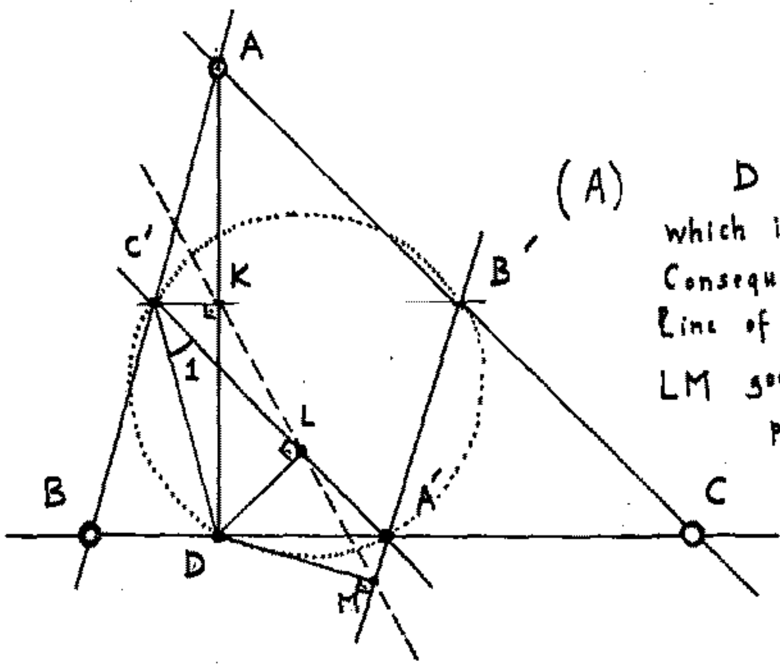
$$=\frac{1}{\Delta^2}\left(4s^2-2s\underbrace{(a+b+c)}_{2s}+a^2+b^2+c^2\right)$$

$$=\frac{a^2+b^2+c^2}{\Delta^2}=4\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)$$

$[B, C]$ and D be the foot of the perpendicular from A onto BC.

(A) Let L, M be the respective feet of the perpendiculars from D onto $C'A'$, $A'B'$. Prove that LM bisects $[A, D]$.

(B) Compute the angle between LM and BC.



(A) D is on the 9-point circle which is the circumcircle of $A'B'C'$. Consequently, LM is the Simson Line of D w.r.t. $A'B'C'$. Therefore LM goes through the foot K of the perpendicular from B onto $B'C'$. K is the midpoint of $[A, D]$.

(B) In $A'B'C'$, LM is the Simson line of D and the Simson line of A' is perpendicular to $B'C'$. Therefore

$$\measuredangle(LM, BC) = \measuredangle(LM, B'C') = \measuredangle(LM, \ell_{A'}) + \frac{\pi}{2} \pmod{\pi}$$

$$= -\measuredangle(C'D, C'A') + \frac{\pi}{2}$$

$$= -(B - C) + \frac{\pi}{2}$$

$$= \frac{\pi}{2} + C - B \pmod{\pi}$$

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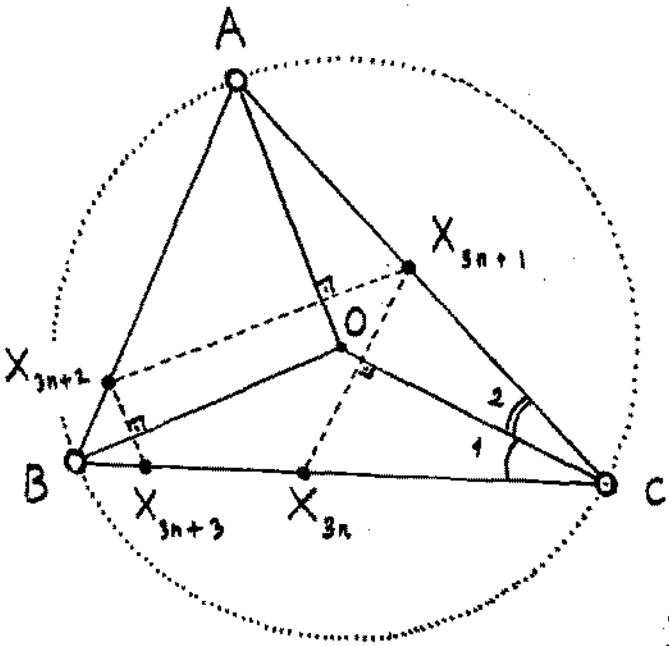
4.

Consider a triangle ABC with circumcenter O. Let X_n be a sequence of points with $X_{3n} \in BC$, $X_{3n+1} \in CA$, $X_{3n+2} \in AB$ such that $OC \perp X_{3n}X_{3n+1}$, $OA \perp X_{3n+1}X_{3n+2}$, $OB \perp X_{3n+2}X_{3n+3}$ for all $n \in \mathbb{Z}$. Prove that $X_n = X_{n+6}$ for all $n \in \mathbb{Z}$.



For convenience we assume BC, CA, AB to be directed so as to give a counterclockwise orientation to ABC. Notice that "1" = $90 - A$, "2" = $90 - B$. We have

$$X_{3n}C \cos(90-A) = CX_{3n+1}\cos(90-B)$$

and
$$CX_{3n+1} = \frac{\sin A}{\sin B} X_{3n}C = \frac{a}{b} X_{3n}C .$$

Therefore
$$X_{3n+1}A = b - CX_{3n+1} = b - \frac{a}{b} X_n C .$$

By the same reasoning
$$X_{3n+2}B = c - \frac{b}{c} X_{3n+1}A = c - \frac{b}{c}\left(b - \frac{a}{b} X_n C\right)$$
$$= c - \frac{b^2}{c} + \frac{a}{c} X_n C .$$

Finally
$$X_{3n+3}C = a - \frac{c}{a} X_{3n+2}B = a - \frac{c}{a}\left(c - \frac{b^2}{c} + \frac{a}{c} X_n C\right)$$
$$= \frac{a^2+b^2-c^2}{a} - X_n C$$ hence $X_{3n+6}C = X_{3n}C$.

Similarly... $\therefore X_{3n} = X_{3n+6}$