SIXTH LECTURE

Surfaces in Euclidean Spaces

DEFINITION 1 : A *surface* of class C^k in $\mathbb{R}^n$ is a map $\mathfrak{q} : \Omega \longrightarrow \mathbb{R}^n$ of class C^k where Ω is an open connected subset of $\mathbb{R}^2$. $\mathfrak{q}$ is said to be *regular* at $(u_0, v_0) \in \Omega$ if $D\mathfrak{q}|_{(u_0,v_0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ is of rank 2. If $n = 3$ this is equivalent to the requirement that

$$\frac{\partial \mathfrak{q}}{\partial u} \times \frac{\partial \mathfrak{q}}{\partial v} \Bigg|_{(u_0,v_0)} \neq \mathbf{0} \quad .$$

$\mathfrak{q}$ is said to be *regular* if it is regular at each point of its domain.

In the following we shall consider smooth and regular surfaces only .

EXAMPLE 1 : Let $\Omega = \mathbb{R} \times (0, \pi)$. Consider $\mathfrak{q} : \Omega \longrightarrow \mathbb{R}^3$ defined by

$$\mathfrak{q}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \ .$$

Observe that $\mathfrak{q}$ traces out the set $\mathbf{S}^2 - \{(0,0,1), (0,0,-1)\} \subseteq \mathbb{R}^3$.

EXAMPLE 2 : Consider $\mathfrak{q} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by

$$\mathfrak{q}(u, v) = (u, v, u^2 + v^2) \quad .$$

Note that this is a *circular paraboloid* . It is obtained by rotating a parabola about its axis.

EXAMPLE 3 : Let Given any $M \in \mathbb{R}^3$ and any independent vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, consider $\mathfrak{q} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by

$$\mathfrak{q}(u, v) = M + u\mathbf{a} + v\mathbf{b} \quad .$$

Observe that $\mathfrak{q}$ traces out a plane in $\mathbb{R}^3$ through the point M and parallel to the subspace spanned by the vectors $\mathbf{a}, \mathbf{b}$.

EXAMPLE 4 : Given $a > b > 0$, consider $\mathfrak{q} : (-1,1) \times \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by

$$\mathfrak{q}(u,\theta) = \left(\left[a + bu\cos\left(\frac{\theta}{2}\right) \right] \cos\theta \ , \left[a + bu\cos\left(\frac{\theta}{2}\right) \right] \sin\theta \ , bu\sin\left(\frac{\theta}{2}\right) \right)$$

Observe that $\mathfrak{q}$ traces out a shape which is known as the *Möbius strip* .

EXAMPLE 5 : Given $a > b > 0$, consider $\mathfrak{q} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by

$$\mathfrak{q}(\theta,\varphi) = ((a + b\cos\varphi)\cos\theta \ , (a + b\cos\varphi)\sin\theta \ , b\sin\theta \ )$$

Observe that $\mathfrak{q}$ traces out a *torus* which is obtained by taking the circle of radius b and center $(a,0,0)$ in the zx-plane and rotating it about the z axis.

EXAMPLE 6 : The purpose of this example is to generalise the examples 1, 2, 5 above. The *surface of revolution* obtained by rotating the curve $x = \varphi(t)$, $z = \psi(t)$ for $t \in J$ in the zx-plane is the surface $\mathfrak{q} : \mathbb{R} \times J \longrightarrow \mathbb{R}^3$ defined by

$$\mathfrak{q}(t,\theta) = (\varphi(t)\cos\theta, \varphi(t)\sin\theta, \psi(t))$$

for each $(t,\theta) \in J \times \mathbb{R}$. The intuitive content becomes clear by noticing that the right hand side is the point with the position vector

$$\mathfrak{q}(t,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi(t) \\ 0 \\ \psi(t) \end{bmatrix} \quad .$$

EXAMPLE 7 : Consider $\mathfrak{q} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by

$$\mathfrak{q}(u,v) = \left(\frac{2u}{1+u^2+v^2} \ , \ \frac{2v}{1+u^2+v^2} \ , \ \frac{1-u^2-v^2}{1+u^2+v^2} \right)$$

Observe that $\mathfrak{q}$ traces out the set $\mathbf{S}^2 - \{(0,0,-1)\} \subseteq \mathbb{R}^3$. $\mathfrak{q}$ establishes a bijection between $\mathbb{R}^2$ and $\mathbf{S}^2 - \{(0,0,-1)\}$ Indeed $\mathfrak{q}^{-1}$ is the two dimensional version of the stereographic projection. For each $(u,v) \in \mathbb{R}^2$, let the straight line joining $(0,0,-1)$ and $(u,v,0)$ intersect $\mathbf{S}^2$ for a second time in $\mathfrak{q}(u,v)$. Clearly

$x : y : 1 + z = u : v : 1$ hence

$$u = \frac{x}{1+z} \quad \text{and} \quad v = \frac{y}{1+z} \quad \text{and} \quad u^2 + v^2 = \frac{x^2+y^2}{(1+z)^2} = \frac{1-z^2}{(1+z)^2} = \frac{1-z}{1+z}$$

consequently

$$z = \frac{1-u^2-v^2}{1+u^2+v^2} \ , \ x = \frac{2u}{1+u^2+v^2} \ , \ y = \frac{2v}{1+u^2+v^2} \quad .$$

DEFINITION 2 : Given a *surface* $\mathfrak{q} : \Omega \longrightarrow \mathbb{R}^3$ the *unit normal field* associated with $\mathfrak{q}$ is a function $\mathbf{n} : \Omega \longrightarrow \mathbb{R}^3$ defined by

$$\mathbf{n} = \frac{\dfrac{\partial \mathfrak{q}}{\partial u} \times \dfrac{\partial \mathfrak{q}}{\partial v}}{\left\| \dfrac{\partial \mathfrak{q}}{\partial u} \times \dfrac{\partial \mathfrak{q}}{\partial v} \right\|} \quad .$$

EXAMPLE 8 : By a simple computation it can be seen that the unit normal field associated with the circular paraboloid described in Example 2 is

$$\mathbf{n}(u,v) = (1 + 4u^2 + 4v^2)^{-1/2} \begin{bmatrix} -2u \\ -2v \\ 1 \end{bmatrix} \quad .$$

DEFINITION 3 : Given a *surface* $\mathfrak{q} : \Omega \longrightarrow \mathbb{R}^3$ with unit normal field $\mathbf{n}$ the *tangent plane* of $\mathfrak{q}$ for $(u,v) = (u_0, v_0) \in \Omega$ is the plane in $\mathbb{R}^3$ through the point $\mathfrak{q}(u_0, v_0)$ and perpendicular to $\mathbf{n}(u_0, v_0)$.

EXAMPLE 10 : The tangent plane to the circular paraboloid at the point $\mathfrak{q}(1,1) = (\ 1\ ,\ 1\ ,\ 2\ )$ is

$$-2(x-1) - 2(y-1) + 1(z-2) = 0 \ .$$

DEFINITION 4 : Given a surface $\mathfrak{q} : \Omega \longrightarrow \mathbb{R}^3$ a curve $\mathfrak{r} : J \longrightarrow \mathbb{R}^3$ is said to *lie on the surface* $\mathfrak{q}$ or said to be a *curve on the the surface* $\mathfrak{q}$ if there exists a curve $\gamma : J \longrightarrow \mathbb{R}^2$ such that $\gamma(J) \subseteq \Omega$ and $\mathfrak{r} = \mathfrak{q} \circ \gamma$.

REMARK 1 : Notice that for any curve $\mathfrak{r} = \mathfrak{q} \circ \gamma : J \longrightarrow \mathbb{R}^3$ on the surface $\mathfrak{q}$,

$$\dot{\mathfrak{r}}(t) = \left. D\mathfrak{q} \right|_{\gamma(t)} (\dot{\gamma}(t))$$

or simply

$$\dot{\mathfrak{r}} = D\mathfrak{q}\ \dot{\gamma}$$

using matrix notation.

REMARK 2 : In the following whenever mention is made of a curve on a surface $\mathfrak{q}$ we shall talk about a curve $\mathfrak{q} \circ \gamma$ directly.

PROBLEMS

I. Let $a \in \mathbb{R}$ be a constant. Prove that the volume of the tetrahedron formed by the three coordinate planes and the tangent plane at a point on the surface

$$\mathsf{q}(u,v) = \quad u \ , \ v \ , \ \frac{a^3}{uv}$$

where $u, v \in \mathbb{R}$ is a constant.

II. Let $a \in \mathbb{R}$ be a constant. Prove that the sum of the squares of the intercepts of the coordinate axes by a tangent plane to the surface

$$\mathsf{q}(u,v) = \quad u^3 \sin^3 v, u^3 \cos^3 v, (a^2 - u^2)^{\frac{3}{2}}$$

where $u, v \in (-a, a)$, is a constant.

III. Prove that every tangent plane to the surface $\mathsf{q} : \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}^3$ defined by

$$\mathsf{q}(u,v) = \quad u \ , \ v \ , \ \frac{u^3 + v^3}{u^2 + v^2}$$

passes through the point $(0,0,0) \in \mathbb{R}^3$.