ELEMENTARY DIFFERENTIAL GEOMETRY

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A collection of sketches in preparation for a set of lecture notes on elementary differential geometry.

TERMINOLOGY, NOTATION AND CONVENTIONS

(1) The elements of $\mathbb{R}^m$ will be regarded as "points" and as such will constitute the basic ingredients of our geometric investigations. As usual, a point $P \in \mathbb{R}^m$ will be represented by means of a horizontal list of its components :

$$P = (a_1, a_2, \cdots, a_m).$$

(2) $\mathbb{R}^m$ will be endowed with its usual vector space structure. In this context the elements of $\mathbb{R}^m$ will be written as "column vectors" or $m \times 1$ matrices . This initiates the ordinary discourse of analytic geometry by identifying a point with its "position vector" :

$$P = (a_1, a_2, \cdots, a_m) \sim \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{R}^m.$$

(3) Characters standing for vectors and matrices will be printed in boldface, matrix notation will be used freely. The set of $m \times n$ matrices with real entries will be denoted by $\mathbb{R}^{m \times n}$, T as superscript will effect transposition. For instance

$$\mathbf{M} = \begin{bmatrix} 1 & 8 & 3 \\ 5 & 6 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

whereas,

$$\mathbf{M}^T = \begin{bmatrix} 1 & 5 \\ 8 & 6 \\ 3 & 2 \end{bmatrix} \in \mathbb{R}^{3\times 2}.$$

(4) $\mathbb{R}^{m\times 1}$ will be identified with $\mathbb{R}^m$. As an important special case, $\mathbb{R}^{1\times 1}$ will be identified with $\mathbb{R}^1 \sim \mathbb{R}$. For instance

$$\mathbf{a} = [a_1 \;\; a_2 \;\; \cdots \;\; a_m]^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{R}^{m\times 1} \sim \mathbb{R}^m.$$

In particular

$$\mathbf{0}_m = \mathbf{0} = [0 \;\; 0 \;\; \cdots \;\; 0]^T = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{m\times 1} \sim \mathbb{R}^m.$$

(5) Given $\mathbf{a} = [a_1 \;\; a_2 \;\; \cdots \;\; a_m]^T$, $\mathbf{b} = [b_1 \;\; b_2 \;\; \cdots \;\; b_m]^T \in \mathbb{R}^m$ their *inner product* is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}.\mathbf{b} = \mathbf{a}^T\mathbf{b} = [a_1 \;\; a_2 \;\; \cdots \;\; a_m] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \sum_{k=1}^{m} a_k b_k \in \mathbb{R}^1.$$

The *norm* $\| \mathbf{a} \|$ of a vector $\mathbf{a}$ is

$$\| \mathbf{a} \| = \langle \mathbf{a}, \mathbf{a} \rangle^{1/2} = (\mathbf{a}.\mathbf{a})^{1/2}$$

(6) Given $k \in \mathbb{Z}$ with $k \geq 0$, an open set $U \subseteq \mathbb{R}^m$ and a function $f : U \longrightarrow \mathbb{R}^n$, f will be said to be of class C^k if it has continuous k th derivative. f is said to be of class C^∞ or simply *smooth* if it is of class C^k for all $k \in \mathbb{Z}$ with $k \geq 0$.

(7) Given $n \in \mathbb{N}$, $\mathbf{S}^n$ denotes sphere of unit radius in $\mathbb{R}^{n+1}$ with center at $\mathbf{0}$. To be precise

$$\mathbf{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \;\; \| \mathbf{x} \| \;\; = \;\; 1\}.$$

(8) Given an open $U \subseteq \mathbb{R}$ and three times differentiable $\alpha : U \longrightarrow \mathbb{R}^n$, $\dot{\alpha}$, $\ddot{\alpha}$, $\dddot{\alpha}$ (rarely) will stand respectively for the derivative, second derivative, third derivative of α. To be precise,

$$\dot{\alpha}(t) = \frac{d\alpha}{dt}(t) \; , \; \ddot{\alpha}(t) = \frac{d^2\alpha}{dt^2}(t) \; , \; \dddot{\alpha}(t) = \frac{d^3\alpha}{dt^3}(t)$$

for each $t \in U$.

(9) Given vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m \in \mathbb{R}^{m\times 1} \sim \mathbb{R}^m$

$$det[\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m]$$

is the determinant of the matrix made up of the vectors in question.

(10) Given vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ their vector product will be denoted by $\mathbf{a} \times \mathbf{b}$.