§ 29. Laurent Series :

Given analytic $f : \Omega \longrightarrow \mathbb{C}$ and $a \in \Omega$, the function $\varphi : \Omega \longrightarrow \mathbb{C}$ defined by ~~$\varphi(z)$~~

$$\varphi(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{if } z \neq a \\ f'(a) & \end{cases}$$

is known to be continuous. Observe that, in fact, φ is analytic, since

$$\varphi(z) = a_1 + a_2(z-a) + \dots + a_n(z-a)^{n-1} + \dots$$

where $a_n = \dfrac{1}{n!} f^{(n)}(a)$ near $z = a$.

Theorem: (The "Laurent Series")

Given analytic $f : \Omega \xrightarrow{\subseteq_{op} \mathbb{C}} \mathbb{C}$ and $a \in \mathbb{C} - \Omega$ and $R > r > 0$ with

$$\overline{B}(a, R) - B(a, r) = \{ z \mid r \leqslant |z-a| \leqslant R \} \subseteq \Omega$$

$$(*) \qquad \left\| \; f(z) = \sum_{n=-\infty}^{+\infty} A_n (z-a)^n \; \right\| \; (*)$$

for $r < |z-a| < R$

~~where~~

where $$A_n = \frac{1}{2\pi i} \int_{\Gamma_{a,\rho}} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta$$

for any ρ ~~such~~ with $r \leq \rho \leq R$.

—— Interlude ——

Explaining the "biinfinite" power series (*)

By $\sum_{n=-\infty}^{+\infty} A_n (z-a)^n$ we understand the sum of

$$\sum_{n=0}^{\infty} A_n (z-a)^n$$ (which will be shown to be convergent for $|z-a| < R$)

and

$$\sum_{n=-\infty}^{-1} A_n (z-a)^n$$ (which will be shown to be convergent for $|z-a| > r$)

—— End of the interlude ——

Proof: Consider any $z \in \Omega$ with $r < |z-a| < R$. As remarked above $\varphi : \Omega \longrightarrow \mathbb{C}$ defined by

$$\varphi(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \text{for } \zeta \neq z \\[2ex] f'(z) & \text{for } \zeta = z . \end{cases}$$

is analytic. Consequently

$$\int_{\Gamma_{a,R}} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = \int_{\Gamma_{a,r}} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta$$

(by the usual "homotopy trick" or by dividing the region between $\Gamma_{a,R}$, $\Gamma_{a,r}$ into star shaped sets!)

Since $\displaystyle\int_{\Gamma_{a,R}} \frac{d\zeta}{\zeta - z} = 2\pi i$, $\displaystyle\int_{\Gamma_{a,r}} \frac{d\zeta}{\zeta - z} = 0$

we obtain

$$2\pi i\, f(z) = \int_{\Gamma_{a,R}} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{\Gamma_{a,r}} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \int_{\Gamma_{a,R}} \frac{f(\zeta)}{\zeta - a - (z - a)}\, d\zeta - \int_{\Gamma_{a,r}} \frac{f(\zeta)}{\zeta - a - (z - a)}\, d\zeta$$

$$= \int_{\Gamma_{a,R}} \frac{f(\zeta)}{\zeta - a} \frac{1}{1 - \frac{z - a}{\zeta - a}}\, d\zeta + \int_{\Gamma_{a,r}} \frac{f(\zeta)}{z - a} \frac{1}{1 - \frac{\zeta - a}{z - a}}\, d\zeta$$

Notice that $\left|\frac{z - a}{\zeta - a}\right| < 1$ in the first integral whereas ~~but~~ $\left|\frac{\zeta - a}{z - a}\right| <$ in the second !

$$= \int_{\Gamma_{a,R}} \frac{f(\zeta)}{\zeta - a} \sum_{n=0}^{\infty} \left(\frac{z-a}{\zeta - a} \right)^n d\zeta + \int_{\Gamma_{a,r}} \frac{f(\zeta)}{z-a} \sum_{n=0}^{\infty} \left(\frac{\zeta - a}{z-a} \right)^n d\zeta$$

$$= \int_{\Gamma_{a,R}} \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - a)^{n+1}} (z-a)^n \, d\zeta + \int_{\Gamma_{a,r}} \sum_{n=0}^{\infty} f(\zeta)(\zeta - a)^{\overset{-N-1}{n}} (z-a)^{\overbrace{-n-1}^{N}} d\zeta$$

As the involved power series are uniformly convergent on any compact set in $\{\zeta \mid |\zeta - a| > |z-a|\}$ and in $\{\zeta \mid |\zeta - a| < |z-a|\}$ we can interchange integration and infinite sums:

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\Gamma_{a,R}} \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta \right) (z-a)^n$$

$$+ \sum_{n=-\infty}^{-1} \left(\frac{1}{2\pi i} \int_{\Gamma_{a,r}} \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta \right) (z-a)^n$$

*

Both these closed curves can be replaced by $\Gamma_{a,\rho}$ for any $\rho > 0$ with $r \leq \rho \leq R$.

§30. Isolated singularities :

We shall be mainly concerned with the Laurent series in the situations where it is possible to take $r > 0$ arbitrarily small.

notation of §29

Given analytic $f : \Omega \longrightarrow \mathbb{C}$, $a \in \mathbb{C} - \Omega$ is called an isolated singularity of f if $\exists\ \varepsilon > 0$ such that

$$\underbrace{B(a,\varepsilon) - \{a\}}_{\text{the "punctured" } \varepsilon\text{-neighbourhood of } a.} \subseteq \Omega .$$

By the results of §29, if $R = \sup\{\rho \mid \overline{B}(a,\rho) - \{a\} \subseteq \Omega\}$ then for any z with $0 < |z-a| < R$

$$f(z) = \sum_{n=-\infty}^{+\infty} A_n (z-a)^n \qquad \text{where} \quad A_n = \frac{1}{2\pi i} \int_{\Gamma_{a,\rho}} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta$$

$\rho >$ any number with the property that $\overline{B}(a,\rho) - \{a\} \subseteq \Omega$.

! any $\rho > 0$ sufficiently small will do!

! Vital detail ! $\underbrace{\sum_{n=-\infty}^{-1} A_n (z-a)^n}_{\text{"Singular part" of } f \text{ at } z=a.}$ is convergent for all $z \in \mathbb{C} - \{a\}$!

Important quantity : For an isolated singularity a of $f : \Omega \to \mathbb{C}$ the number

$$A_{-1} = \frac{1}{2\pi i} \int_{\Gamma_{a,\rho}} f(\zeta)\, d\zeta$$

is called the **residue** of f at $z = a$.

~~$0 < \rho$ with~~ any

any $\rho > 0$ with $\overline{B}(a,\rho) - \{a\} \subseteq \Omega$

Notation :

$$\boxed{\operatorname{Res}_{z=a} \left(f(z) \right)}$$

"how singular is a singularity ?"

Isolated singularities are classified according to the "size" of the "singular part" :

1) $z = a$ is called a removable singularity if the singular part is identically zero. This means

that f is the restriction of an analytic $\tilde{f} : \underbrace{\Omega \cup \{a\}}_{\text{open}} \longrightarrow \mathbb{C}$ to Ω !

Intuitively, this means that a is really a part of the domain of f ; it has been excluded only as a result of some technicality :

Typically, in the presence of an analytic $F : \Lambda \underset{op}{\subseteq} \mathbb{C} \to \mathbb{C}$, $a \in \Lambda$ is a removable singularity of $f : \Lambda - \{a\} \longrightarrow \mathbb{C}$ defined by $f(z) = \dfrac{F(z) - F(a)}{z - a}$ for $z \in \Lambda - \{a\}$.

<u>Examples:</u> $z = 1$ is a removable singularity of

$$f(z) = \frac{\log z}{z - 1} . \qquad \left(f : \mathbb{C}_{-} - \{1\} \longrightarrow \mathbb{C} \right)$$

$z = 0$ is a r.s. of $f(z) = \dfrac{\sin z}{z} \quad (f : \mathbb{C} - \{0\} \to \mathbb{C})$

$z = 0$ " " " " " $f(z) = \dfrac{e^z - 1}{z} \quad \left(f : \mathbb{C} - \{0\} \longrightarrow \mathbb{C} \right)$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots \qquad = \frac{1}{1!} + \frac{z}{2!} + \frac{z^2}{3!} + \ldots$$

2) $z = a$ is called a <u>pole (of order n)</u> if the singular part is a polynomial in $\dfrac{1}{z - a}$ (of order n).

A pole of order 1 is referred to as a <u>simple pole</u> .

A removable singularity may be thought of as a pole of order 0 .

Example :

$$\frac{e^{z}}{z^4} = \underbrace{\frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{2!\, z^2} + \frac{1}{3!\, z}}_{\text{singular part}} + \frac{1}{4!} + \frac{1}{5!} z + \dots$$

singular part $\longrightarrow$ pole of order 4

$$\operatorname{Res}_{z=0}\left(\frac{e^{z}}{z^4}\right) = \frac{1}{6}.$$

Clearly a is a pole of order m iff (for $f(z)$)

a is a removable singularity of $(z-a)^m f(z)$.

~~Using~~ ~~&~~ If a is a pole of order m

$$f(z) = \frac{A_{-m}}{(z-a)^m} + \dots + \frac{A_{-1}}{z-a} + A_0 + A_1(z-a) + \dots \qquad A_{-m} \neq 0$$

hence

$$(z-a)^m f(z) = A_{-m} + A_{-m+1}(z-a) + \dots + A_{-1}(z-a)^{m-1} + \dots$$

and

$$\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right] = (m-1)!\, A_{-1} + m!\, A_0 (z-a) + \dots$$

for $z \neq a$.

$\therefore$

$$\boxed{\operatorname{Res}_{z=a}\left(f(z)\right) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]}$$

3) $z = a$ is called an <u>essential</u> isolated singular point is a is <u>not</u> a pole!

PROBLEMS (13)

1. In each one of the following, $z = 0$ is a singular point. Classify it . Find its order if it is a pole. Compute the residue in each case.

$$\frac{z+5}{z^2(z-i)}\,,\quad \frac{(z+5)\cosh(z^2)}{z^2(z-i)}\,,\quad \frac{(z+5)\sinh(z^2)}{z^2(z-i)}\,,\quad \frac{(z+5)}{z^2(z-i)\cosh(z^5)}$$

$$\frac{(z+5)(e^{z^2}-1)}{z^2(z-i)}\,,\quad \frac{(z+5)\cosh(z^2)}{z^2(z-i)(e^{z^2}-1)}\,,\quad \frac{(z+5)\sinh(z^2)}{z^2(z-i)(e^{z^2}-1)}$$

2. Show that

(A) $\operatorname{Res}_{z=2}\left(\dfrac{z^5}{z^3-8}\right) = \dfrac{8}{3}$

(B) $\operatorname{Res}_{z=i}\left(\dfrac{3z^2+7}{z^2+1}\right) = -2i$

(C) $\operatorname{Res}_{z=1}\left(\exp\left(\dfrac{1}{z-1}\right)\right) = 1$

(D) $\operatorname{Res}_{z=(-1+i\sqrt{3})/2}\left(\dfrac{e^{2z}+e^z+2}{z^2+z+1}\right) = \dfrac{e^{-2+i\sqrt{3}}+e^{-1+i\sqrt{3}/2}+2}{i\sqrt{3}}$

(E) $\operatorname{Res}_{z=0}\left(z\cos\left(\dfrac{1}{z}\right)\right) = -\dfrac{1}{2}$

(F) $\operatorname{Res}_{z=0}\left(\dfrac{\cot(z)}{z^4}\right) = -\dfrac{1}{45}$

(G) $\operatorname{Res}_{z=0}\left(\dfrac{\sinh(z)}{z^4(1-z^2)}\right) = \dfrac{7}{6}$

(H) $\operatorname{Res}_{z=i}\left(\dfrac{\log(z)}{(z^2+1)^2}\right) = \dfrac{\pi+2i}{8}$

(I) $\operatorname{Res}_{z=0}\left(\dfrac{z^{1/2}}{z^2+1}\right) = \dfrac{1-i}{2\sqrt{2}}$

3. Prove that

(A) $\displaystyle\int_\gamma \frac{dz}{z^3(z+3)^2}\,dz = \frac{2}{27}\pi i$ where $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ with $\gamma(t) = 2e^{it}$

(B) $\displaystyle\int_\gamma \frac{dz}{z^3(z+3)^2}\,dz = -\frac{2}{27}\pi i$ where $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ with $\gamma(t) = -4 + 3e^{it}$

(C) $\displaystyle\int_\gamma \frac{z^3 e^{1/z}}{1+z^3}\,dz = -\frac{2\pi e}{3} i$ where $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ with $\gamma(t) = -1 + e^{it}$