

§ 29. Laurent Series :

Given analytic $f: \Omega \rightarrow \mathbb{C}$ and $a \in \Omega$, the function $\varphi: \Omega \rightarrow \mathbb{C}$ defined by ~~$\varphi(z)$~~

$$\varphi(z) = \begin{cases} \frac{f(z) - f(a)}{z - a} & \text{if } z \neq a \\ f'(a) & \end{cases}$$

is ^{known to be} continuous. Observe that, in fact, φ is analytic, since

$$\varphi(z) = a_1 + a_2(z-a) + \dots + a_n(z-a)^{n-1} + \dots$$

where $a_n = \frac{1}{n!} f^{(n)}(a)$ near $z = a$.

Theorem: (The "Laurent Series")

Given analytic $f: \Omega \xrightarrow{\mathbb{C}} \mathbb{C}$ and $a \in \mathbb{C} - \Omega$
and $R > r > 0$ with

$$\overline{B}(a, R) - B(a, r) = \{z \mid r \leq |z-a| \leq R\} \in \Omega$$

$$(*) \quad \left\| \begin{aligned} f(z) &= \sum_{n=-\infty}^{+\infty} A_n (z-a)^n \end{aligned} \right\| (*)$$

for $r < |z-a| < R$
where

where

$$A_n = \frac{1}{2\pi i} \int_{\Gamma_{r,p}} \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta$$

for ^{any} p ~~with~~ with $r < p < R$.

~~~~~ Interlude ~~~~~

Explaining the "biiinfinite" power series (\*)

By  $\sum_{n=-\infty}^{+\infty} A_n (z-a)^n$  we understand the sum of

$$\sum_{n=0}^{\infty} A_n (z-a)^n \quad \left( \text{which will be shown to be convergent for } |z-a| < R \right)$$

and

$$\sum_{n=-\infty}^{-1} A_n (z-a)^n \quad \left( \text{which will be shown to be convergent for } |z-a| > r \right)$$

~~~~~ End of the interlude ~~~~~

Consider any $z \in \Omega$ with $r < |z-a| < R$.

Proof: As remarked above $\varphi : \Omega \rightarrow \mathbb{C}$ defined

by

$$\varphi(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \text{for } \zeta \neq z \\ f'(z) & \text{for } \zeta = z. \end{cases}$$

is analytic. Consequently

$$\int_{\Gamma_{a,R}} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta = \int_{\Gamma_{a,r}} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta$$

Since $\int_{\Gamma_{a,R}} \frac{d\zeta}{\zeta - z} = 2\pi i$, $\int_{\Gamma_{a,r}} \frac{d\zeta}{\zeta - z} = 0$

(by the usual "homotopy trick" or by dividing the region between $\Gamma_{a,R}$, $\Gamma_{a,r}$ into star shaped sets!)

we obtain

$$\begin{aligned} 2\pi i f(z) &= \int_{\Gamma_{a,R}} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\Gamma_{a,r}} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \int_{\Gamma_{a,R}} \frac{f(\zeta)}{\zeta - a - (z-a)} d\zeta - \int_{\Gamma_{a,r}} \frac{f(\zeta)}{\zeta - a - (z-a)} d\zeta \\ &= \int_{\Gamma_{a,R}} \frac{f(\zeta)}{\zeta - a} \frac{1}{1 - \frac{z-a}{\zeta - a}} d\zeta + \int_{\Gamma_{a,r}} \frac{f(\zeta)}{z-a} \frac{1}{1 - \frac{\zeta - a}{z-a}} d\zeta \end{aligned}$$

Notice that ~~$|\frac{z-a}{\zeta - a}| < 1$~~ in the first integral whereas ~~but~~ $|\frac{\zeta - a}{z-a}| < 1$ in the second!

$$\begin{aligned}
 &= \int_{\Gamma_{a,R}} \frac{f(\zeta)}{\zeta-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{\zeta-a}\right)^n d\zeta + \int_{\Gamma_{a,r}} \frac{f(\zeta)}{z-a} \sum_{n=0}^{\infty} \left(\frac{\zeta-a}{z-a}\right)^n d\zeta \\
 &= \int_{\Gamma_{a,R}} \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta-a)^{n+1}} (z-a)^n d\zeta + \int_{\Gamma_{a,r}} \sum_{n=0}^{\infty} f(\zeta) (\zeta-a)^{n-N-1} (z-a)^{N-n-1} d\zeta
 \end{aligned}$$

As the involved power series are uniformly convergent on any compact set in $\{\zeta \mid |\zeta-a| \geq |z-a|\}$ and in $\{\zeta \mid |\zeta-a| < |z-a|\}$ we can interchange integration and infinite sums:

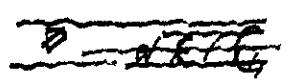
$$\begin{aligned}
 f(z) &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\Gamma_{a,R}} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta \right) (z-a)^n \\
 &\quad + \sum_{n=-\infty}^{-1} \left(\frac{1}{2\pi i} \int_{\Gamma_{a,r}} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta \right) (z-a)^n
 \end{aligned}$$

Both these closed curves can be replaced by $\Gamma_{a,p}$ for any $p > 0$ with $r \leq p \leq R$.

§30. Isolated singularities:

We shall be ^{mainly} concerned with the Laurent series in the situations where it is possible to take $r > 0$ arbitrarily small.

notation of §29



Given analytic $f : \Omega \rightarrow \mathbb{C}$, $a \in \mathbb{C} - \Omega$ is called an isolated singularity of f if $\exists \epsilon > 0$ such that

$$B(a, \epsilon) \setminus \{a\} \subseteq \Omega.$$

the "punctured" ϵ -neighbourhood of a .

By the results of §29, if $R = \sup \{ \rho \mid \overline{B}(a, \rho) - \{a\} \subseteq \Omega \}$ then for any z with $0 < |z - a| < R$

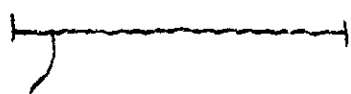
$$f(z) = \sum_{n=-\infty}^{+\infty} A_n (z-a)^n \quad \text{where} \quad A_n = \frac{1}{2\pi i} \int_{\Gamma_{a, \rho}} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta$$

! any $\rho > 0$ sufficiently small will do!

$\rho > 0$ any number with the property that $\overline{B}(a, \rho) - \{a\} \subseteq \Omega$.

! Vital detail ! $\sum_{n=-\infty}^{-1} A_n (z-a)^n$ is convergent for all $z \in \mathbb{C} - \{a\}$!

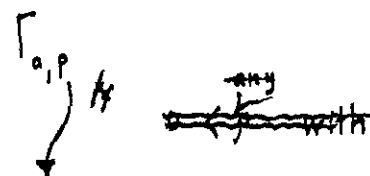
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"Singular part" of f
at $z=a$.

Important quantity : For an isolated singularity a of $f: \Omega \rightarrow \mathbb{C}$

the number

$A_{-1} = \frac{1}{2\pi i} \int_{\Gamma_{a,p}} f(z) dz$ is called the residue of f at $z=a$.



any $p > 0$ with $\overline{B}(a,p) - \{a\} \subseteq \Omega$

Notation :

$$\boxed{\text{Res}_{z=a} (f(z))}$$

"how singular is a singularity?"

Isolated singularities are classified according to the "size" of the "singular part" :

- 1) $z=a$ is called a removable singularity if the singular part is identically zero. This means

that f is the restriction of an analytic $\tilde{f} : \underbrace{\Omega \cup \{a\}}_{\text{open}} \rightarrow \mathbb{C}$
to Ω !

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Intuitively, this means that a is really a part of ~~the~~ the domain of f ; it has been excluded only as a result of some technicality:

Typically, in the presence of an analytic $F : \Lambda \subseteq \mathbb{C} \rightarrow \mathbb{C}$
 $a \in \Lambda$ is a removable singularity of $f : \Lambda - \{a\} \rightarrow \mathbb{C}$
defined by $f(z) = \frac{F(z) - F(a)}{z - a}$ for $z \in \Lambda - \{a\}$.

Examples: $z = 1$ is a removable singularity of

$$f(z) = \frac{\log z}{z - 1} \quad \left(f : \mathbb{C}_- - \{1\} \rightarrow \mathbb{C} \right)$$

$z = 0$ is a r.s. of $f(z) = \frac{\sin z}{z} \quad (f : \mathbb{C} - \{0\} \rightarrow \mathbb{C})$
 $z = 0$ " " " " " " $f(z) = \frac{e^z - 1}{z} \quad (f : \mathbb{C} - \{0\} \rightarrow \mathbb{C})$
 $= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$ $= \frac{1}{1!} + \frac{z}{2!} + \frac{z^2}{3!} + \dots$

2) $z = a$ is called a pole (of order n) if

the singular part is a polynomial in $\frac{1}{z-a}$ (of order n).

A pole of order 1 is referred to as a simple pole.

A removable singularity may be thought of as a pole of order 0.

Example:

$$\frac{e^z}{z^4} = \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{2!z^2} + \frac{1}{3!z} + \frac{1}{4!} + \frac{1}{5!}z + \dots$$

singular part \rightarrow pole of order 4

$$\text{Res}_{z=0} \left(\frac{e^z}{z^4} \right) = \frac{1}{6}$$

Clearly a is a pole of order m iff

a is a removable singularity of $(z-a)^m f(z)$.

Using \mathcal{P} If a is a pole of order m

$$f(z) = \frac{A_{-m}}{(z-a)^m} + \dots + \frac{A_{-1}}{z-a} + A_0 + A_1(z-a) + \dots$$

hence

$$(z-a)^m f(z) = A_{-m} + A_{-m+1}(z-a) + \dots + A_{-1}(z-a)^{m-1} + \dots$$

and

$$\frac{d}{dz} \frac{d}{dz} \dots \frac{d}{dz} \left[(z-a)^m f(z) \right] = (m-1)! A_{-1} + m! A_0 (z-a) + \dots$$

for $z \neq a$.

$$\text{Res}_{z=a} (f(z)) = \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$$

3) $z=a$ is called an essential isolated singular point
 if a is not a pole!

PROBLEMS (13)

1. In each one of the following, $z = 0$ is a singular point. Classify it. Find its order if it is a pole. Compute the residue in each case.

$$\frac{z+5}{z^2(z-i)}, \quad \frac{(z+5)\cosh(z^2)}{z^2(z-i)}, \quad \frac{(z+5)\sinh(z^2)}{z^2(z-i)}, \quad \frac{(z+5)}{z^2(z-i)\cosh(z^5)}$$

$$\frac{(z+5)(e^{z^2}-1)}{z^2(z-i)}, \quad \frac{(z+5)\cosh(z^2)}{z^2(z-i)(e^{z^2}-1)}, \quad \frac{(z+5)\sinh(z^2)}{z^2(z-i)(e^{z^2}-1)}$$

2. Show that

$$(A) \operatorname{Res}_{z=2} \left(\frac{z^5}{z^3-8} \right) = \frac{8}{3}$$

$$(B) \operatorname{Res}_{z=i} \left(\frac{3z^2+7}{z^2+1} \right) = -2i$$

$$(C) \operatorname{Res}_{z=1} \left(\exp \left(\frac{1}{z-1} \right) \right) = 1$$

$$(D) \operatorname{Res}_{z=(-1+i\sqrt{3})/2} \left(\frac{e^{2z}+e^z+2}{z^2+z+1} \right) = \frac{e^{-2+i\sqrt{3}}+e^{-1+i\sqrt{3}/2}+2}{i\sqrt{3}}$$

$$(E) \operatorname{Res}_{z=0} \left(z \cos \left(\frac{1}{z} \right) \right) = -\frac{1}{2}$$

$$(F) \operatorname{Res}_{z=0} \left(\frac{\cot(z)}{z^4} \right) = -\frac{1}{45}$$

$$(G) \operatorname{Res}_{z=0} \left(\frac{\sinh(z)}{z^4(1-z^2)} \right) = \frac{7}{6}$$

$$(H) \operatorname{Res}_{z=i} \left(\frac{\log(z)}{(z^2+1)^2} \right) = \frac{\pi+2i}{8}$$

$$(I) \operatorname{Res}_{z=0} \left(\frac{z^{1/2}}{z^2+1} \right) = \frac{1-i}{2\sqrt{2}}$$

3. Prove that

$$(A) \int_{\gamma} \frac{dz}{z^3(z+3)^2} dz = \frac{2}{27}\pi i \quad \text{where } \gamma: [0, 2\pi] \rightarrow \mathbb{C} \text{ with } \gamma(t) = 2e^{it}$$

$$(B) \int_{\gamma} \frac{dz}{z^3(z+3)^2} dz = -\frac{2}{27}\pi i \quad \text{where } \gamma: [0, 2\pi] \rightarrow \mathbb{C} \text{ with } \gamma(t) = -4 + 3e^{it}$$

$$(C) \int_{\gamma} \frac{z^3 e^{1/z}}{1+z^3} dz = -\frac{2\pi e}{3} i \quad \text{where } \gamma: [0, 2\pi] \rightarrow \mathbb{C} \text{ with } \gamma(t) = -1 + e^{it}$$