

PROBLEMS (10)

1. Evaluate the following integrals :

$$(A) \int_{\gamma} (3z^2 - 5z + i) dz \quad \text{where } \gamma : [0, \pi/2] \rightarrow \mathbb{C} \text{ with } \gamma(t) = ie^{-it}$$

$$(B) \int_{\gamma} (e^z - 4z) dz \quad \text{where } \gamma : [0, \pi] \rightarrow \mathbb{C} \text{ with } \gamma(t) = e^{it}$$

$$(C) \int_{\gamma} \frac{dz}{z} \quad \text{where } \gamma : [0, 2\pi] \rightarrow \mathbb{C} \text{ with } \gamma(t) = 7 + 3 \cos t + 2i \sin t$$

$$(F) \int_{\gamma} e^z \cos z dz \quad \text{where } \gamma : [0, 1] \rightarrow \mathbb{C} \text{ with } \gamma(t) = (1 - t^5)\pi + it^3$$

$$(C) \int_{\gamma} \frac{dz}{z^2 + 4} \quad \text{where } \gamma : [0, 2\pi] \rightarrow \mathbb{C} \text{ with } \gamma(t) = 1 + i + e^{it}$$

$$(C) \int_{\gamma} \frac{dz}{z^2 + 4z + 1} \quad \text{where } \gamma : [0, 2\pi] \rightarrow \mathbb{C} \text{ with } \gamma(t) = i + e^{it}$$

§ 27. Logarithms

Given $b \in \mathbb{C}$, $a \in \mathbb{C}$ is called a logarithm of b if $e^a = b$. Since the exponential function is nowhere zero, there is no logarithm of 0. On the other hand if $b \neq 0$, the equation $e^a = b$ holds iff

$$|e^a| = e^{\operatorname{Re}(a)} = |b| \quad \text{and} \quad e^{i \operatorname{Im}(a)} = \frac{b}{|b|}.$$

usual log on \mathbb{R} .

The first equation has the obvious unique solution $\operatorname{Re}(a) = \log(|b|)$ and the second equation has infinitely many solutions

$$\operatorname{Im}(a) = \theta \in \mathbb{R} \quad \text{satisfying}$$

$$\cos \theta = \frac{\operatorname{Re}(b)}{|b|}, \quad \sin \theta = \frac{\operatorname{Im}(b)}{|b|}.$$

Clearly, if $a \in \mathbb{C}$ is a logarithm of $b \in \mathbb{C}$ then $a' \in \mathbb{C}$ is a logarithm of b iff $a' - a \in 2\pi i \mathbb{Z}$. To sum up: Each $b \in \mathbb{C} - \{0\}$ has ~~infinitely~~ countably many logarithms which differ from one another by an integer multiple of $2\pi i$.

Given $\Omega \subseteq \mathbb{C}$, we say " $\varphi : \Omega \rightarrow \mathbb{C}$ " is a logarithmic function on Ω if φ is a continuous function that satisfies $e^{\varphi(z)} = z$ for all $z \in \Omega$. Clearly there exists no logarithmic function on Ω if $0 \in \Omega$!

Thus we shall always take $\Omega \subseteq \mathbb{C} - \{0\}$.

By the inverse function theorem applied to $\exp : \mathbb{C} \cong \mathbb{R}^2 \rightarrow \mathbb{C}$ it is readily seen that \exp is locally bijective with smooth local inverse. This shows that a logarithmic function $\psi(z) = u(x,y) + iv(x,y)$ is not only smooth, but since

$$e^u \cos v = x, \quad e^u \sin v = y$$

$$e^u \frac{\partial u}{\partial x} \cos v - e^u \sin v \frac{\partial v}{\partial x} = 1$$

$$e^u \frac{\partial u}{\partial y} \cos v - e^u \sin v \frac{\partial v}{\partial y} = 0$$

$$e^u \frac{\partial u}{\partial x} \sin v + e^u \cos v \frac{\partial v}{\partial x} = 0$$

$$e^u \frac{\partial u}{\partial y} \sin v + e^u \cos v \frac{\partial v}{\partial y} = 1$$

and

$$\frac{\partial u}{\partial x} = e^{-u} \cos v, \quad \frac{\partial v}{\partial x} = -e^{-u} \sin v, \quad \frac{\partial u}{\partial y} = e^{-u} \sin v$$

$$\frac{\partial v}{\partial y} = e^{-u} \cos v.$$

which clearly obey the Cauchy-Riemann equations, $\Psi(z)$ is an ~~analytic~~ analytic function of z and

$$1 = (z)' = (e^{\Psi(z)})' = \underset{z}{e^{\Psi(z)}} \Psi'(z)$$

hence $\Psi'(z) = \frac{1}{z}$ for all $z \in \Omega$.

** \therefore A logarithmic function $\varphi : \Omega \subseteq \mathbb{C} - \{0\} \rightarrow \mathbb{C}$ is an antiderivative of $\frac{1}{z}$ on Ω .

Remark: It is impossible to define a logarithmic function on $\mathbb{C} - \{0\}$ since $\int \frac{1}{z} dz = 2\pi i \neq 0$

($\gamma : [0, 2\pi] \rightarrow \mathbb{C}, \gamma(t) = e^{it}$!) and $\frac{1}{z}$ has no

antiderivative on $\mathbb{C} - \{0\}$!

** Conversely, ~~if~~ each antiderivative $\varphi : \Omega \subseteq \mathbb{C} - \{0\} \rightarrow \mathbb{C}$ of $\frac{1}{z}$ on Ω differs from a logarithmic function by a constant: Indeed

$$\frac{d}{dz} \frac{e^{\varphi(z)}}{z} = \frac{e^{\varphi(z)} \cdot \varphi'(z) \cdot z - e^{\varphi(z)} \cdot 1}{z^2} = 0$$

hence $\exists c \neq 0$ with $e^{\varphi(z)} = cz$ for all $z \in \Omega$. 74

choosing $c \in \mathbb{C} \setminus \{0\}$ with $e^c = c$ it is seen that

$\Psi(z) = \varphi(z) - c$ is a logarithmic function on Ω as

$$e^{\Psi(z)} = e^{\varphi(z) - c} = c \cdot z \cdot c^{-1} = z$$

for all $z \in \Omega$.

Thus, on any simply connected open subset of $\mathbb{C} - \{0\}$ we can define a logarithmic function.

In particular, logarithmic functions on $\mathbb{C}_- = \mathbb{C} - (-\infty, 0]$
the so called "complex plane cut along the negative x-axis"

have a simple description: The "principal logarithm"

$\log : \mathbb{C}_- \longrightarrow \mathbb{C}$ is defined by

$$\log(z) = \underbrace{\log(|z|)}_{\text{the ordinary log on } \mathbb{R}!} + \theta i$$

where $\theta \in \mathbb{R}$ is uniquely determined by $e^{i\theta} = \frac{z}{|z|}$ and $-\pi < \theta < \pi$.

Every other logarithmic function on \mathbb{C}_- is of the form

$$\log(z) + 2\pi i \cdot n$$

for some $n \in \mathbb{Z}$.

On any set $\Omega \subseteq \mathbb{C} - \{0\}$ on which a logarithmic function ψ is defined we can define the α th power of z

where $\alpha \in \mathbb{C}$, by $z^\alpha = e^{\alpha \psi(z)}$. In particular the

principal α th power

z^α on \mathbb{C}_- is defined

by $z^\alpha \stackrel{\text{def}}{=} e^{\alpha \log(z)}$.

PROBLEMS (11)

1. Prove that

$$(A) \quad \log(ei) = 1 + \frac{\pi}{2}i$$

$$(B) \quad \log(1+i) = \frac{1}{2}\log 2 + \frac{\pi}{4}i$$

$$(C) \quad \log(e - ei) = 1 + \frac{1}{2}\log 2 - \frac{\pi}{4}i$$

2. Compute

$$(A) \quad \log\left(-\frac{\sqrt{3}}{2} - i\right), \quad (B) \quad \log(e^2 - e^2i), \quad (C) \quad \log(\log(e^{\pi/4} + e^{\pi/4}i))$$

3. Compute

$$\lim_{n \rightarrow \infty} \log\left(-1 + \frac{1}{n}i\right) \quad \text{and} \quad \lim_{n \rightarrow \infty} \log\left(-1 - \frac{1}{n}i\right)$$

4. Prove that

$$\operatorname{Re}((1+i)^i) = e^{-\pi/4} \cos(\log \sqrt{2})$$

5. Evaluate the following integrals :

$$(A) \quad \int_{\gamma} \frac{dz}{z} \quad \text{where } \gamma : [-1, 1] \rightarrow \mathbb{C} \text{ with } \gamma(t) = -2t^2 + 1 + it$$

$$(B) \quad \int_{\gamma} \frac{dz}{z} \quad \text{where } \gamma : [-1, 1] \rightarrow \mathbb{C} \text{ with } \gamma(t) = i - 2ti$$

$$(C) \quad \int_{\gamma} z^{1/2} dz \quad \text{where } \gamma : [-1, 1] \rightarrow \mathbb{C} \text{ with } \gamma(t) = -t^2 + 1 + it$$

$$(D) \quad \int_{\gamma} (\log z) dz \quad \text{where } \gamma : [-1, 1] \rightarrow \mathbb{C} \text{ with } \gamma(t) = -2t^2 + 1 + it$$

$$(E) \quad \int_{\gamma} (\log z)^2 dz \quad \text{where } \gamma : [0, 1] \rightarrow \mathbb{C} \text{ with } \gamma(t) = 1 - t + it$$

$$(F) \quad \int_{\gamma} \frac{dz}{1+z^2} \quad \text{where } \gamma : [0, 1] \rightarrow \mathbb{C} \text{ with } \gamma(t) = 1 - t^2 + (1+i)t$$