Augustin Louis Cauchy 1789-1857
Edouard Jean Baptiste Goursat 1858-1936

§26. The Cauchy-Goursat theorem:

An open connected set $\Omega \subseteq \mathbb{C}$ is called simply connected if every differentiable $f : \Omega \longrightarrow \mathbb{C}$ has an antiderivative.

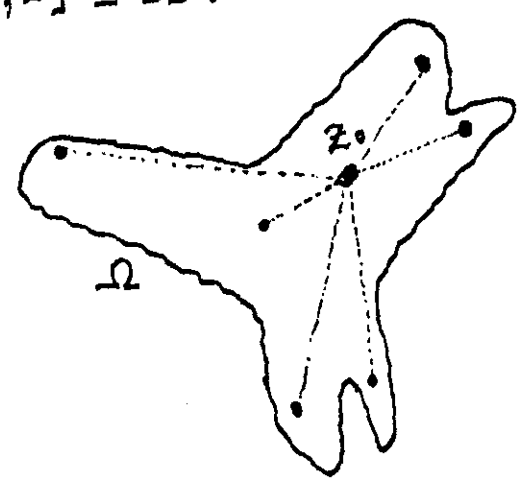
It turns out that (although I will ~~not~~ neither prove ~~neither~~ nor make use of the result!) $\Omega \subseteq \mathbb{C}$ is simply connected iff any point (if any!) outside Ω is the initial point of a curve extending to infinity.

The purpose of the present section is to point out a large and easily recognisable family of simply connected sets.

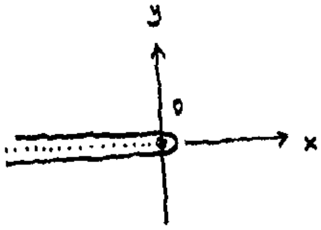
Given $A, B \in \mathbb{C}$, ~~let~~ the line segment $[A, B]$ is the set

$$[A,B] = \left\{ (1-t)A + tB \;\middle|\; t \in [0,1] \right\}$$

Definition: Ω is said to be star shaped if $\exists z_0 \in \Omega$ such that $\forall z \in \Omega$, $[z_0, z] \subseteq \Omega$.



<u>Examples:</u> All convex sets are star shaped. In particular, $\mathbb{C}$ itself and open and closed balls



$$B(a,R) = \{z \in \mathbb{C} \mid |z-a| < R\}$$

$$\overline{B}(a,R) = \{z \in \mathbb{C} \mid |z-a| \leqslant R\} .$$

$\mathbb{C}_- = \mathbb{C} - (-\infty, 0]$ the "complex plane cut along the negative real axis," is star shaped.

<u>Remark:</u> (Proof left to the student!) If Ω is starshaped, then for each $\varepsilon > 0$, the ε-neighbourhood $\cancel{\Omega_\varepsilon}$

$$\Omega_\varepsilon = \bigcup_{a \in \Omega} B(a, \varepsilon) = \{z \mid \exists a \in \Omega,\ |z-a| < \varepsilon\}$$

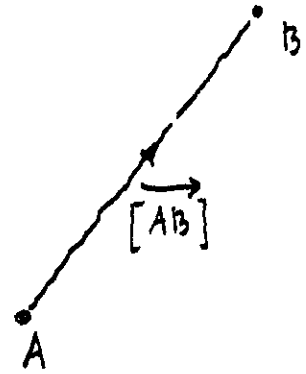
of Ω is star shaped, too.

<u>Theorem:</u> A starshaped set is simply connected.

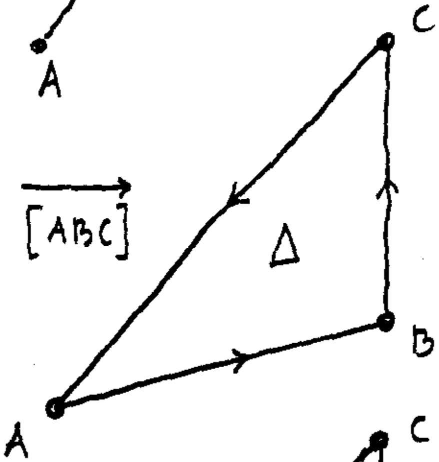
The proof will be given after a few preliminaries and a technical lemma:

Given $A, B \in \mathbb{C}$ let $\overrightarrow{[AB]}$ be the directed line segment joining A to B. To be precise, $\overrightarrow{[AB]}$ is the curve $\gamma : [0,1] \longrightarrow \mathbb{C}$ defined by $\gamma(t) = (1-t)A + tB$.

Given $A, B, C \in \mathbb{C}$ let $\overrightarrow{[ABC]}$ be the directed triangle

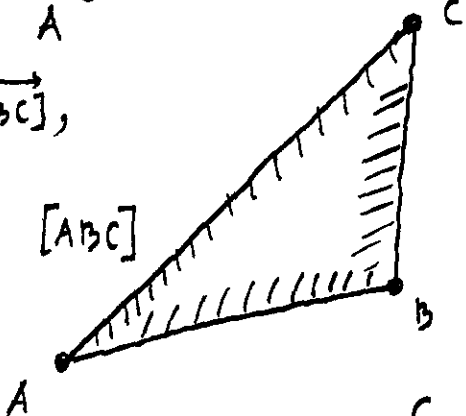


$$\overrightarrow{[ABC]} = \overrightarrow{[AB]} * \overrightarrow{[BC]} * \overrightarrow{[CA]} .$$

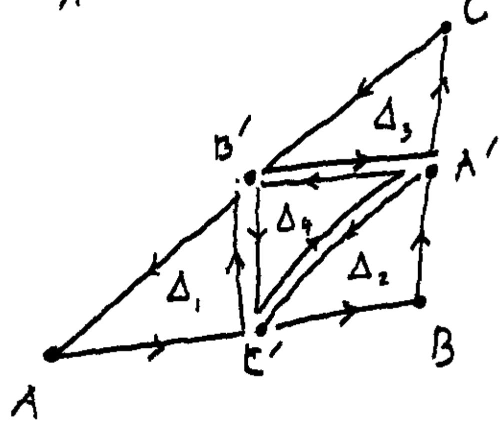


Let $[ABC]$ denote the points in or on $\overrightarrow{[ABC]}$, that is

$$[ABC] = \{ \lambda A + \mu B + \nu C \mid \lambda, \mu, \nu \geqslant 0,\ \lambda + \mu + \nu = 1 \} .$$



Let $A' = \frac{1}{2}(B+C)$, $B' = \frac{1}{2}(C+A)$, $C' = \frac{1}{2}(A+B)$ be the respective midpoints of $[BC]$, $[CA]$, $[AB]$. The <u>medial decomposition</u> of $\overrightarrow{[ABC]} = \Delta$



is the set

$$\Delta_1 = \overrightarrow{[AC'B']} ,\quad \Delta_2 = \overrightarrow{[BA'C']} ,\quad \Delta_3 = \overrightarrow{[CB'A']} ,\quad \Delta_4 = \overrightarrow{[A'B'C']}$$

of directed triangles.

Lemma: ("The Cauchy-Goursat theorem")

For any directed triangle $\overrightarrow{[ABC]}$ with $[ABC] \subseteq \Omega$ and any differentiable $f : \Omega \underset{op}{\subseteq} \mathbb{C} \longrightarrow \mathbb{C}$

$$\int_{\overrightarrow{[ABC]}} f(z)\, dz = 0 .$$

Proof: For any triangle $\Delta = \overrightarrow{[XYZ]}$ ~~that~~ with $[XYZ] \subseteq \Omega$

$$\int_{\Delta} f(z)\, dz = \sum_{k=1}^{4} \int_{\Delta_k} f(z)\, dz$$

Δ_k the medial decomposition of Δ !

hence

$$\left| \int_{\Delta} f(z)\, dz \right| \leq \sum_{k=1}^{4} \left| \int_{\Delta_k} f(z)\, dz \right| .$$

We choose Δ' from among Δ_k, $k = 1, 2, 3, 4$ to be one with the property

$$\left| \int_{\Delta} f(z)\, dz \right| \leq 4 \left| \int_{\Delta'} f(z)\, dz \right| .$$

Now turning our attention to $\Delta = \overrightarrow{[ABC]}$ we define Δ_n inductively, by

$$\underset{\overrightarrow{[A_n B_n C_n]}}{\Delta_n} \quad \Delta_0 = \Delta \quad , \quad \Delta_{n+1} = \Delta_n' .$$

Let L be the length of $\Delta = \overrightarrow{[ABC]}$. $\exists$ unique $a \in \Omega$ such that

$$\{a\} = \bigcap_{n=0}^{\infty} [A_n B_n C_n]$$

(Remember that $[A_n B_n C_n]$ is a compact set; its diameter tending to zero as $n \to \infty$. Apply the "finite intersection property"!)

Since f is differentiable, $\exists$ ~~a~~ neighbourhood N of $a \in \Omega$ and a function $K : N \to \mathbb{C}$ such that

$$\frac{K(z)}{z-a} \longrightarrow 0 \quad \text{as} \quad z \longrightarrow 0$$

and

$$f(z) = f(a) + f'(a)(z-a) + K(z) .$$

$\therefore$ For <u>any</u> $\varepsilon > 0$, $\exists\, n$ s.t. $[A_n B_n C_n] \subseteq N$ and $|K(z)| \leqslant \varepsilon |z-a|$, and

$$\left| \int_{\Delta} f(z)\,dz \right| \leqslant 4^n \left| \int_{\Delta_n} f(z)\,dz \right|$$

$$= 4^n \left| \underbrace{\int_{\Delta_n} \left[f(a) + f'(a)(z-a) \right] dz}_{=0} + \int_{\Delta_n} K(z)\,dz \right|$$

$$= 4^n \left| \int_{\Delta_n} K(z)\,dz \right| \leqslant \cancel{4^n \int |K(z)|}\ 4^n \cdot \frac{L}{2^n} \cdot \varepsilon \frac{L}{2^n} = L^2 \varepsilon$$

arbitrary!

Proof of the theorem: Choose $z_0 \in \Omega$ which can be joined to all points in Ω. Define $F : \Omega \longrightarrow \mathbb{C}$ by

$$F(z) = \int_{\overrightarrow{[z_0, z]}} f(\zeta)\, d\zeta$$

Claim: F is differentiable and $F' = f$! Indeed for any $z \in \Omega$,

$$\frac{1}{\delta}\Big[F(z+\delta) - F(z)\Big] = \frac{1}{\delta}\left[\int_{\overrightarrow{[z_0, z+\delta]}} f(\zeta)\, d\zeta - \int_{\overrightarrow{[z_0, z]}} f(\zeta)\, d\zeta\right]$$

$$= \frac{1}{\delta}\left[\int_{\overrightarrow{[z, z_0]}} f(\zeta)\, d\zeta + \int_{\overrightarrow{[z_0, z+\delta]}} f(\zeta)\, d\zeta\right] = \frac{1}{\delta}\int_{\overrightarrow{[z, z+\delta]}} f(\zeta)\, d\zeta$$

The Lemma!

$$= \frac{1}{\cancel{\delta}}\int_0^1 f(z + t\delta)\cdot \cancel{\delta}\, dt \longrightarrow f(z)$$

by the reasoning in the theorem in §25!