§ 25. Path independence :

Revisiting the two examples of § 24 it is found that the integrals of $f(z) = z$ along γ_1, γ_2 joining $0 \in \mathbb{C}$ to $1+i \in \mathbb{C}$ are the same. This is no coincidence! Integrating $f(z) = z$ along <u>any</u> curve joining 0 to 1 would give the same result : Indeed for any $\gamma : [a,b] \longrightarrow \mathbb{C}$ with $\gamma(a) = 0$, $\gamma(b) = i$

$$\int_\gamma z\,dz = \int_a^b \gamma(t)\,\dot{\gamma}(t)\,dt = \frac{1}{2}\gamma(t)^2\Big|_{t=a}^{t=b}$$

$$= \frac{1}{2}\left[\gamma(b)^2 - \gamma(a)^2\right] = \frac{1}{2}\left[(1+i)^2 - 0^2\right] = i\,.$$

It is easily seen that this is a direct consequence of the fact that $\left(\frac{1}{2}z^2\right)' = z$!

Terminology: $F : \Omega \longrightarrow \mathbb{C}$ is said to be <u>an</u> <u>antiderivative</u> of $f : \Omega \longrightarrow \mathbb{C}$ if F is differentiable and $F' = f$.

(If f has an antiderivative, then it has infinitely many which differ from one another by a constant)

$f : \Omega \longrightarrow \mathbb{C}$ is said to have <u>path independent integrals</u> if for any $P, Q \in \Omega$ and $\gamma_1 : [a,b] \longrightarrow \Omega$, $\gamma_2 : [c,d] \longrightarrow \Omega$ with $\gamma_1(a) = \gamma_2(c)$, $\gamma_1(b) = \gamma_2(d)$

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz\,.$$

Generalising the procedure employed above, it is easily seen that <u>a function with an antiderivative has path independent integrals</u>:

Indeed, if $F' = f$ then

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\dot\gamma(t)\,dt = \int_a^b F'(\gamma(t))\,\dot\gamma(t)\,dt$$

$$= \Big[F(\gamma(t))\Big]_{t=a}^{b}$$

$$= F(\gamma(b)) - F(\gamma(a))$$

$$= F(Q) - F(P) \qquad [*]$$

which depends only on P, Q and is entirely independent of the choice of γ.

<u>Terminology:</u> A curve $\gamma : [a,b] \longrightarrow \mathbb{C}$ is said to be <u>closed</u> if $\gamma(b) = \gamma(a)$.

$*$: <u>Revisiting $*$</u>: If f has an antiderivative, $\int_\gamma f(z)\,dz = 0$ for any <u>closed</u> γ.

In ~~real~~ analysis on $\mathbb{R}$ every continuous function has antiderivative. This is not true in ~~an~~ complex analysis and deciding about the circumstances under which a continuous function does have an antiderivative leads us to embryonic forms of "analysis situs"

$\downarrow$

topology $\longrightarrow$ Homotopy & homology !

Theorem: Given a continuous $f : \Omega \underset{op}{\subseteq} \mathbb{C} \longrightarrow \mathbb{C}$ the following are equivalent:

(i) f has an antiderivative.

(ii) $\displaystyle\int_\gamma f(z)\,dz = 0$ for every closed γ in Ω

(iii) f has path independent integrals...

Proof: (i) $\Longrightarrow$ (ii): Already done.

(ii) $\Longrightarrow$ (iii): ~~Since~~ Consider γ_1, γ_2 with the same starting and ending points: $\gamma_1 * \gamma_2^{-1}$ is a closed curve and

$$\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2^{-1}} f(z)\,dz = \int_{\gamma_1 * \gamma_2^{-1}} f(z)\,dz = 0.$$

(iii) $\Longrightarrow$ (i): (Assume Ω path connected!)

Pick and fix $z_0 \in \Omega$. Define $F : \Omega \longrightarrow \mathbb{C}$ by

$$F(z) = \int_\gamma f(\zeta)\,d\zeta$$

γ being any path joining z_0 to z. F is well-defined by path-independence of integrals of f.

Claim: F is differentiable and $F' = f$: Indeed

$$0 \neq \delta \qquad \frac{1}{\delta}\left[F(z+\delta) - F(z)\right] = \frac{1}{\delta}\left[\int_{\gamma_1} f(\zeta)\,d\zeta - \int_{\gamma_2} f(\zeta)\,d\zeta\right]$$

γ_1 joining z_0 to $z+\delta$

γ_2 joining z_0 to z

$$= \frac{1}{\delta}\left[\int_{\gamma_2^{-1}} f(\zeta)\,d\zeta + \int_{\gamma_1} f(\zeta)\,d\zeta\right] = \frac{1}{\delta}\int_{\gamma_2^{-1} * \gamma_1} f(\zeta)\,d\zeta$$

$\gamma_2^{-1} * \gamma_1$ joining z to $z+\delta$.

By path independence of integrals of f, we can replace $\gamma_2^{-1} * \gamma_1$ with the line segment (sufficiently small δ) λ joining z to $z+\delta$. Put $\lambda(t) = (1-t)z + t(z+\delta)$

$$= z + t\delta, \quad t \in [0,1]:$$

$$\frac{1}{\delta}\left(F(z+\delta) - F(z)\right) = \frac{1}{\delta}\int_{\lambda} f(\zeta)\,d\zeta \qquad u, v : \Omega \to \mathbb{R}$$

$f = u + vi$

$$= \frac{1}{\cancel{\delta}}\int_0^1 f(\underbrace{z + t\delta}_{\lambda(t)})\,\underbrace{\cancel{\delta}}_{\dot\lambda(t)}\,dt$$

$$= \int_0^1 u(z + t\delta)\,dt + \left(\int_0^1 v(z+t\delta)\,dt\right) i$$

MVT for integrals! $= u(z + t^{*}\delta) + v(z + t^{**}\delta)\, i$ where $t^{*}, t^{**} \in [0,1]$

$$\longrightarrow u(z) + v(z)\, i = f(z)$$

<u>Remark</u> : on $\overset{\Omega}{\overset{\parallel}{\mathbb{C}}} - \{0\}$, $f(z) = 1/z$ has <u>no antiderivative</u>

since

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\cancel{e^{it}}}\, i\cancel{e^{it}}\,dt = 2\pi i \neq 0 .$$

γ closed !

$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$

$\gamma(t) = e^{ti}$

For $n \in \mathbb{Z}$, $n \neq -1$

In contrast : $\dfrac{z^{n+1}}{n+1}$ is an antiderivative of z^n

on $\mathbb{C}$ if $n \geqslant 0$

on $\mathbb{C} - \{0\}$ if $n \leqslant -2$.

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