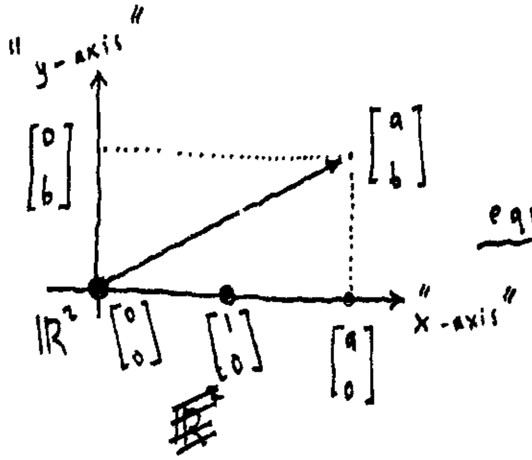
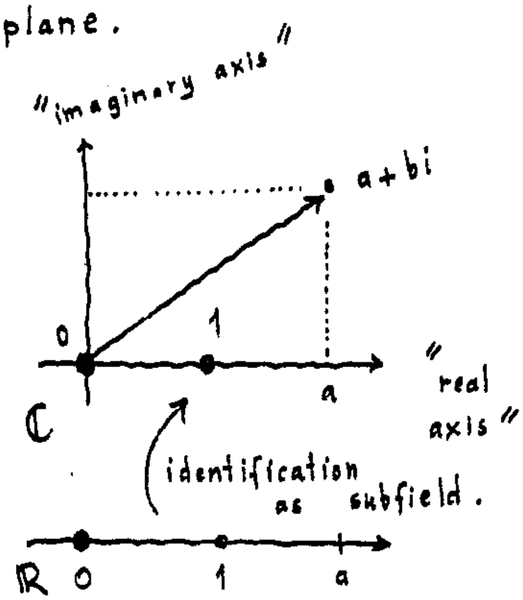
§5. The Argand Diagram :

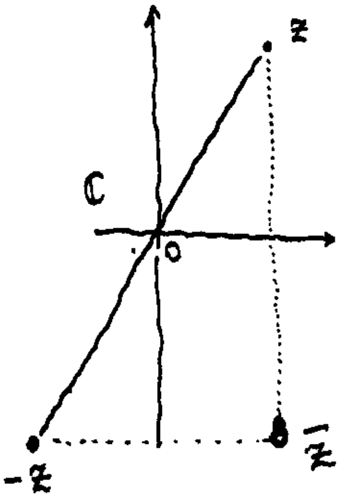
This is the traditional name attached to $\mathbb{C}$ when $\mathbb{C} \sim \mathbb{R}^2$ is identified with the Euclidean plane.

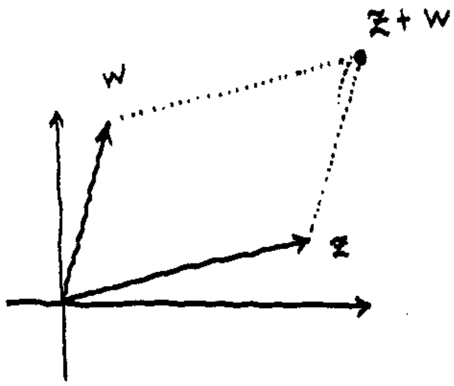


equivalently



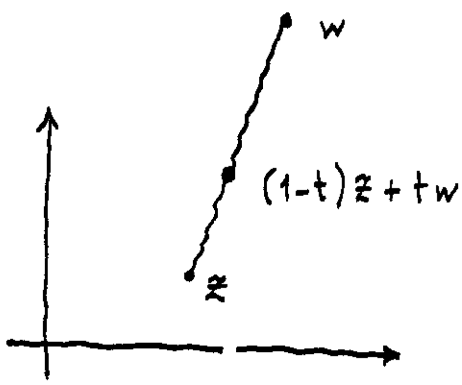
Simplest ideas :





The line segment $[z, w]$:

$$\stackrel{\text{def}}{=} \left\{ (1-t)z + tw \;\middle|\; t \in [0,1] \right\}$$



Euclidean geometry is the geometry of the usual inner product $\langle \cdot , \cdot \rangle$ on $\mathbb{R}^2 \sim \mathbb{C}$:

$$\left\langle \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\sim\, x+iy = z} , \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\sim\, x'+iy' = z'} \right\rangle = xx' + yy' = \mathrm{Re}(\bar{z} z')$$

§6. Modulus of a complex number :

Since we can define inner product on $\mathbb{C}$ we can define "norm", too. Only it is traditionally referred to as "modulus". ($\longrightarrow$ "moduli")

$$\overset{\overset{x+yi}{\|}}{|z|}^2 = \langle z, z \rangle = \mathrm{Re}(\underbrace{\bar{z} z}_{\text{real}}) = \bar{z} z = x^2 + y^2 .$$

Clearly $|z| = 0$ iff $z = 0$.

Revisit §2 : $z^{-1} = \dfrac{\bar{z}}{|z|^2}$ for $z \neq 0$.

(T)

Important : $\boxed{|zw| = |z||w|}$

Proof:

$$|zw|^2 = \overline{zw}\, zw = \bar{z}\bar{w}\, zw = \bar{z}z\, \bar{w}w = |z|^2|w|^2 .$$

(vital observation for number theory: "the two squares problem"

$$(x^2+y^2)(a^2+b^2) = (xa-yb)^2 + (xb+ya)^2$$

$\exists$ similar formulae with $n = 4, 8$.

Important : $\boxed{\mathrm{Re}(\bar{z}w) \leqslant |z||w|}$ "The Cauchy-Schwarz-Bunyakowski inequality"

reduces to equality iff $z \cdot w = 0$ or $\exists\ \lambda > 0$ s.t. $w = \lambda z$

Important : $\boxed{|z+w| \leqslant |z| + |w|}$ "The triangle inequality"

Proof: $|z+w|^2 = \overline{(z+w)}(z+w) = |z|^2 + |w|^2 + 2\mathrm{Re}(\bar{z}w)$

$\leqslant (|z|+|w|)^2$

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PROBLEMS (1)

1. Express the following complex numbers in the form $a + bi$ where $a, b \in \mathbb{R}$.

$$\frac{1}{3+4i} \quad , \quad \frac{(2+3i)(3+2i)}{1-i} \quad , \quad \frac{1-i}{(2+3i)(3+2i)} \ .$$

2. Compute the moduli of

$$3+4i \quad , \quad (2+3i)(3+2i) \quad , \quad \frac{1-i}{(2+3i)(3+2i)} \ .$$

3. For any $z \in \mathbb{C}$, prove that $\mathrm{Im}(iz) = \mathrm{Re}(z)$ and $\mathrm{Re}(iz) = -\mathrm{Im}(z)$.

4. In each of the following cases describe and sketch the set of complex numbers z which satisfy the given relation :

(a) $|z-1| = |z-3|$
(b) $|z-1| = |z-i|$
(c) $\mathrm{Re}((4+i)z+6) = 0$
(d) $\mathrm{Re}(z^2) = 4$
(e) $\mathrm{Im}(z^2) = 8$
(f) $|z^2-1| = 0$
(g) $|z-i| = \mathrm{Re}(z)$

5. Prove that

$$|z|^2 + a^2 = |z+a|^2 - 2\mathrm{Re}(az)$$

for any $a \in \mathbb{R}$, and $z \in \mathbb{C}$.

6. Prove that

$$|z|^2 + 2\mathrm{Re}(bz) = |z+\overline{b}|^2 - |b|^2$$

for any $b, z \in \mathbb{C}$.

7. Given $c \in \mathbb{C} - \{1\}$ with $|c| = 1$, prove that

$$|\frac{z-c}{cz-1}| = 1$$

iff $z \in \mathbb{R}$.

§7. Lines and Circles :

Distance between z & w : $d(z,w) \overset{\text{or}}{=} \text{dist}(z,w) \overset{\text{def}}{=} |z-w|$

A Line $\perp$ to $\begin{bmatrix} A \\ B \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: $Ax + By + C = 0$

$$\Big\downarrow \; x = \frac{z+\bar{z}}{2} \,,\; y = \frac{z-\bar{z}}{2i}$$

$$\boxed{\bar{b} z + b \bar{z} + c = 0}$$

a line $\perp$ to b

where $b = \cancel{A+Bi} \in \mathbb{C}$
$c = \cancel{2C} \in \mathbb{R}$

The circle of center (x_0, y_0) and radius $R > 0$: where $A \neq 0$, $A \in \mathbb{R}$

$$A\left[(x-x_0)^2 + (y-y_0)^2 - R^2\right] = 0$$

$$\downarrow$$

$$A(\bar{z}-\bar{z}_0)(z-z_0) - AR^2 = 0$$

$$\downarrow$$

$$A|z|^2 - A(z_0 \bar{z} + \bar{z}_0 z) + \underbrace{A|z_0|^2 - AR^2}_{c} = 0$$

$a \leftarrow A$
$b \leftarrow -Az_0$

$$\boxed{a z \bar{z} + \bar{b} z + b \bar{z} + c = 0}$$

the circle of
center $-b/a$
radius $\frac{|b|^2 - ac}{a^2}$

where $a, c \in \mathbb{R}$, $b \in \mathbb{C}$ such that $a \neq 0$, $|b|^2 - ac > 0$

PROBLEMS (2)

1. Prove that the points z_1, z_2, $z_3 \in \mathbb{C}$ are collinear iff

$$det \begin{bmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_2 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{bmatrix} = 0 \ .$$

2. Prove that the perpendicular bisector of the line segment with endpoints $b, c \in \mathbb{C}$ is

$$(b-c)\bar{z} + (\bar{b}-\bar{c})z - (b\bar{b} - c\bar{c}) = 0 \ .$$

3. Prove that the point w is the reflection of $z \in \mathbb{C}$ in the line

$$\beta\bar{z} + \bar{\beta}z + \gamma = 0$$

iff

$$\beta\bar{w} + \bar{\beta}z + \gamma = 0 \ .$$

4. Prove that the area Δ of the triangle with vertices $a, b, c \in \mathbb{C}$ is

$$\Delta = det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ \bar{a} & \bar{b} & \bar{c} \end{bmatrix} \ .$$

5. Prove that the lines $a\bar{z}+\bar{a}z+b=0$ and $a'\bar{z}+\bar{a}'z+b'=0$ and $a''\bar{z}+\bar{a}''z+b''=0$ are concurrent iff

$$det \begin{bmatrix} a & \bar{a} & b \\ a' & \bar{a}' & b' \\ a'' & \bar{a}'' & b'' \end{bmatrix} = 0 \ .$$

6. Prove that the points z_1, z_2, z_3, z_4 lie on the same circle or line iff

$$det \begin{bmatrix} 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 \\ \overline{z}_1 & \overline{z}_2 & \overline{z}_3 & \overline{z}_4 \\ z_1\overline{z}_1 & z_2\overline{z}_2 & z_3\overline{z}_3 & z_4\overline{z}_4 \end{bmatrix} = 0 \ .$$

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7. Given distinct $a, b \in \mathbb{C}$ and $\lambda \in \mathbb{R}$, $\lambda \neq 1$, prove that the equation

$$\left|\frac{z-a}{z-b}\right| = \lambda$$

represents a circle whereof the center lies on the line containing a, b. What happens if $\lambda = 1$?