## §23. The exponential function & elementary functions : 

is the function $\exp : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$\boxed{\underset{x+iy}{\exp(\overset{\shortparallel}{z})} \overset{\text{def}}{=} e^{x} \underbrace{(\cos y + i \sin y)}_{\text{" } E(y) \text{ "}}}$$

Henceforth we write $\exp(it) = e^{it}$ instead of $E(t)$, $t \in \mathbb{R}$.

~~The~~ ( $\exp(z)$ is also written as $e^{z}$ ! )

By the partial converse of the Cauchy-Riemann equation applied to the real & imaginary parts

$$u = e^{x} \cos y$$

$$v = e^{x} \sin y$$

it is seen that $e^{z}$ is differentiable everywhere and

$$(e^{z})' = e^{z}.$$

**Simplest properties :**

Thus $\exp : \mathbb{C} \longrightarrow \mathbb{C} - \{0\}$ is a group homomorphism

group w.r.to addition

group w.r.to multiplication !

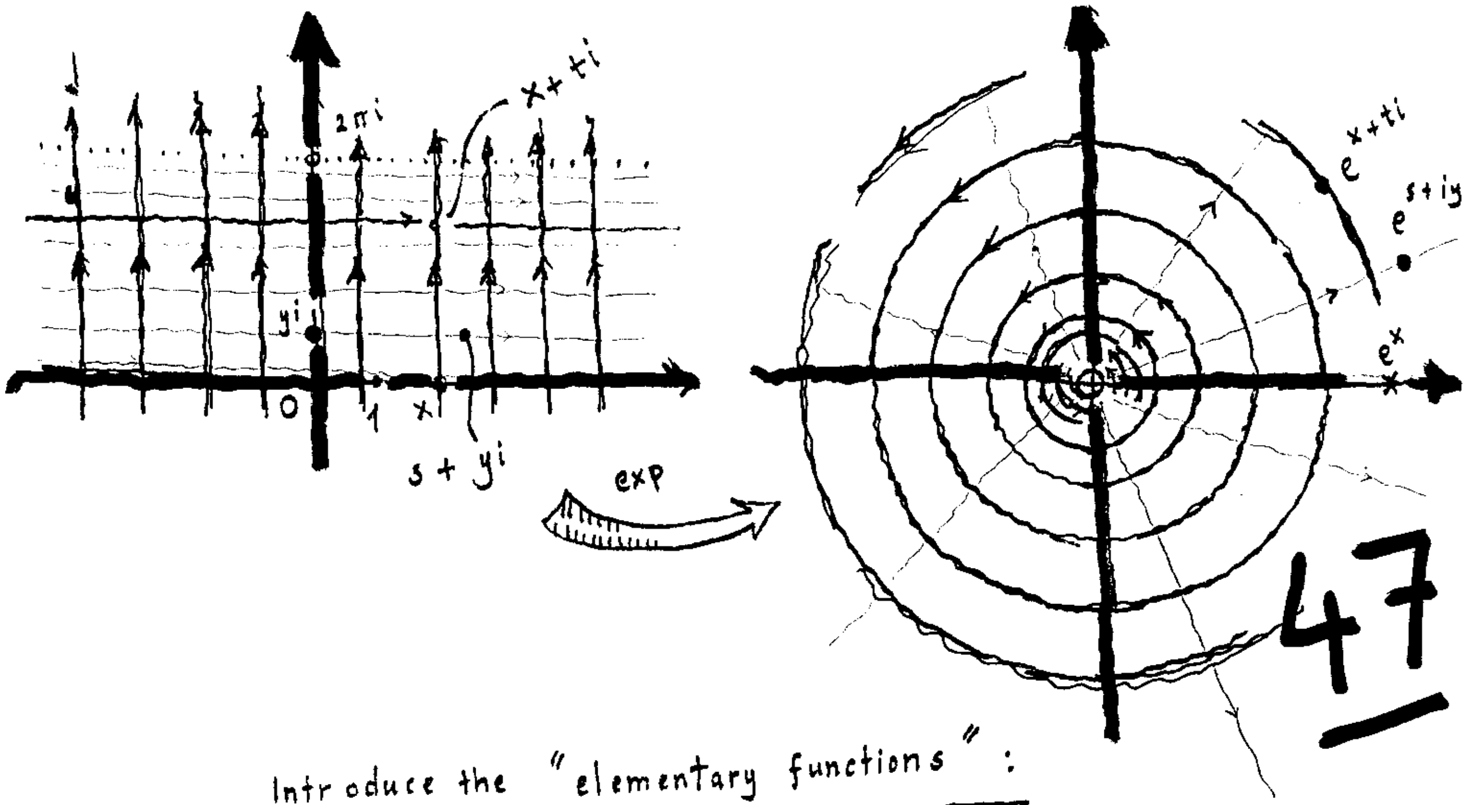
1° $e^{a+b} = e^{a} e^{b}$, $e^{0} = 1$. $\forall a, b \in \mathbb{C}$.

2° $|e^{z}| = e^{\mathrm{Re}(z)}$.

3° $\forall z$ $e^{z} \neq 0$ ( Indeed, $|e^{z}| = e^{\mathrm{Re}(z)} \neq 0$ )

4° $e^{a+2\pi i} = e^{a}$.

Equivalenty $e^{z} = 1$ iff $z \in 2\pi i \mathbb{Z}$.






Introduce the "elementary functions":

$$\cosh(z) = \frac{e^{z}+e^{-z}}{2} \qquad \cos(z) = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sinh(z) = \frac{e^{z}-e^{-z}}{2} \qquad \sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} \qquad \operatorname{sech}(z) = \frac{1}{\cosh(z)} \qquad \tan(z) = \frac{\sin(z)}{\cos(z)} \qquad \sec(z) = \frac{1}{\cos(z)}$$

$$\coth(z) = \frac{1}{\tanh(z)} \qquad \operatorname{csch}(z) = \frac{1}{\sinh(z)} \qquad \cot(z) = \frac{1}{\tan(z)} \qquad \csc(z) = \frac{1}{\sin(z)}$$

Observe: 1° $\sin(z) = 0$ iff $z \in \cancel{2}\pi\mathbb{Z}$

$\cos(z) = 0$ iff $z \in \pi(\mathbb{Z}+\frac{1}{2})$

2° $\sin(iz) = i\sinh(z)$

$\cos(iz) = \cosh(z)$

3° Upon restriction to $\mathbb{R}$, All these function reduce to the ordinary ones!

4° The usual derivatives: $\sin'(z) = \cos(z)$, $\cos'(z) = -\sin(z)$

$\sinh'(z) = \cosh(z)$, $\cosh'(z) = \sinh(z)$.

5° The usual formulae :

$$\sin(a+b) = \sin a \cos b + \sin b \cos a \quad \forall\, a, b \in \mathbb{C}.$$

$$\sinh(a+b) = \sinh(a)\cosh(b) + \sinh(b)\cosh(a)$$

etc.

6° "Pythagoras - Minkowski"

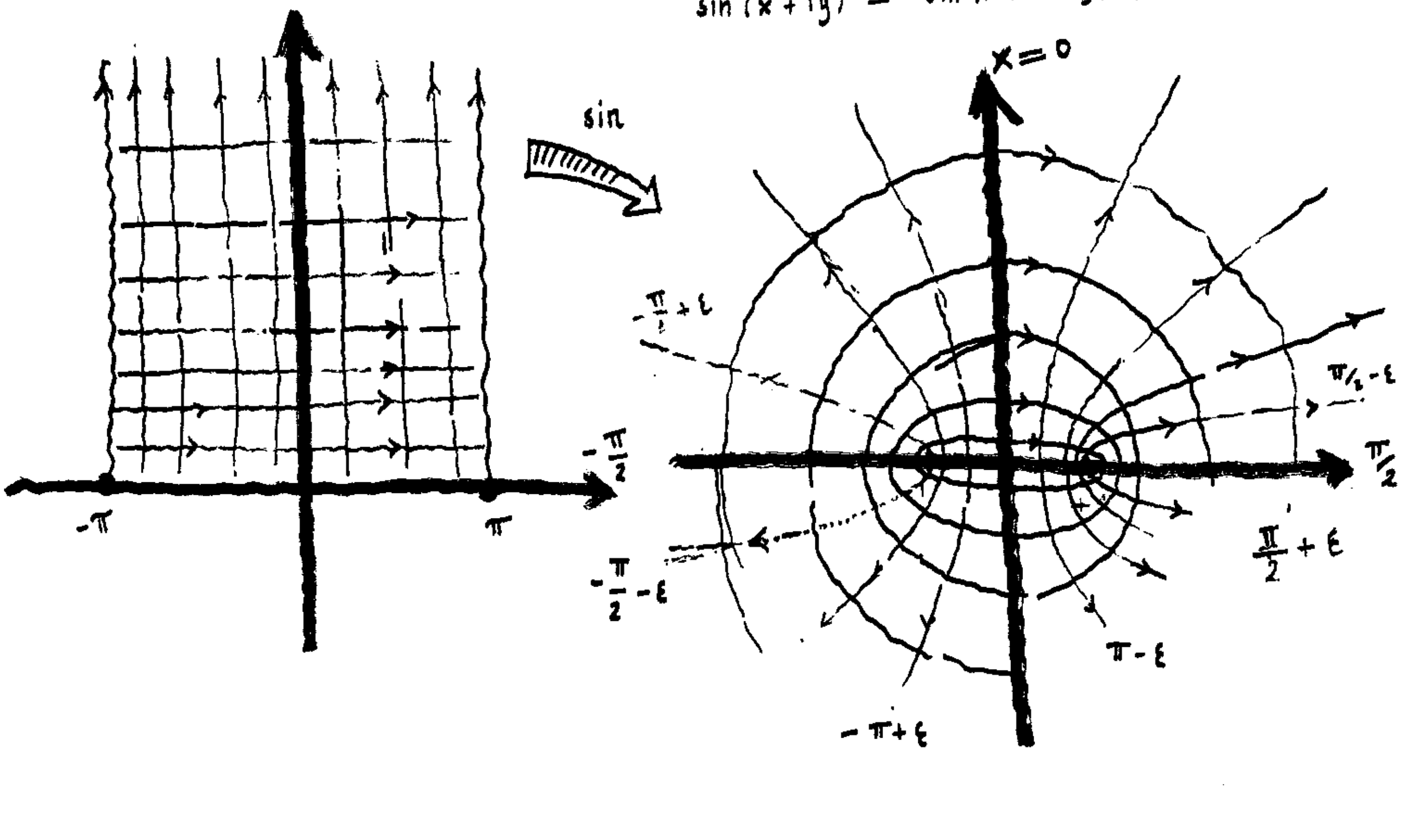
$$\cos^2(z) + \sin^2(z) = 1$$

$$\cosh^2(z) - \sinh^2(z) = 1.$$

7° Warning ! No longer $|\sin(z)| \leqslant 1$ !

Graph of $\sin : \mathbb{C} \longrightarrow \mathbb{C}$ in view of

$$\sin(x+iy) = \sin x \cosh(y) + i \sinh(y) \cos x$$



PROBLEMS (8)

**1.** Express the following quantities in the form $a + bi$ with $a, b \in \mathbb{R}$ :

(A) $\dfrac{\exp(1+3\pi i)}{\exp(-1+i(\pi/2))}$ , (B) $\sin(2i)$ , (C) $\cos(1-i)$ , (D) $\sinh(1+\pi i)$

**2.** Prove the following relations :

(A) $\exp\left(\dfrac{2+\pi i}{4}\right) = (1+i)\sqrt{\dfrac{e}{2}}$

(B) $|\exp(2z+i) + \exp(iz^2)| \leq e^{2x} + e^{-2xy}$

(C) $|\exp(z^2)| \leq \exp|z^2|$

(D) $|\exp(i\exp\bar{z})| = \exp(e^x \sin y)$

**3.** (A) Prove that

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y$$

(B) Find all solutions of the equation

$$\cosh z = \frac{1}{2} \quad .$$

**4.** Prove that

(A) $\lim\limits_{x\longrightarrow\infty} \cos(x+ix) = \lim\limits_{x\longrightarrow\infty} \sin(x+ix) = \infty$

(B) $|\cos(x+iy)| \leq e^y$ if $y \geq 0$ .

(C) $|\sin(x+iy)| \leq e^y$ if $y \geq 0$ .

50

**5.** Compute $f'(z)$ where

$$\text{(A)}\quad f(z) = \exp(\pi z^2)\ , \quad \text{(B)}\quad \sin\left(\frac{2i}{z^4}\right)\ , \quad \text{(C)}\quad \frac{\sinh^3(z+1) + 5z}{\tan z^4}\ .$$

**6.** (A) Prove that

$$|\exp(z) - 1| \leq \exp(|z|) - 1 \leq |z| \exp(|z|)$$

for all $z \in \mathbb{C}$ .

(B) Prove that

$$|\exp(z) - 1| \leq 2|z|$$

for all $z \in \mathbb{C}$ with $|z| \leq 1$ .

## § 23. Complex valued functions of a real variable :

are functions of the form

$$\varphi : J \subseteq \mathbb{R} \longrightarrow \mathbb{C}$$

typically $[a,b]$ or $(a,b)$ where $a, b$ are possibly $\pm\infty$.

So $\varphi = \alpha + i\beta$ where $\alpha, \beta : J \longrightarrow \mathbb{R}$.

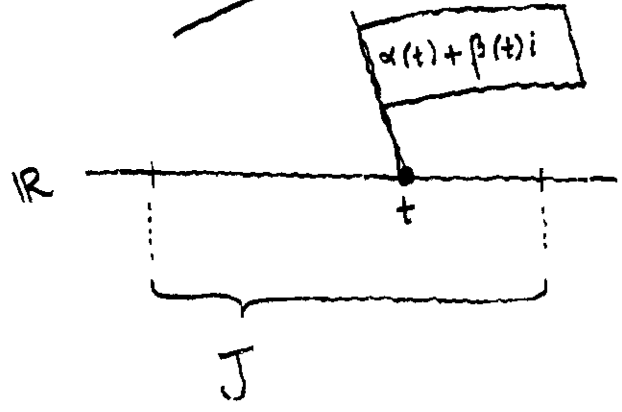
Example:

$J = \mathbb{R}$

$$\varphi(t) = \underbrace{(t + t^2)}_{\alpha(t)} + \underbrace{e^t}_{\beta(t)}\, i$$

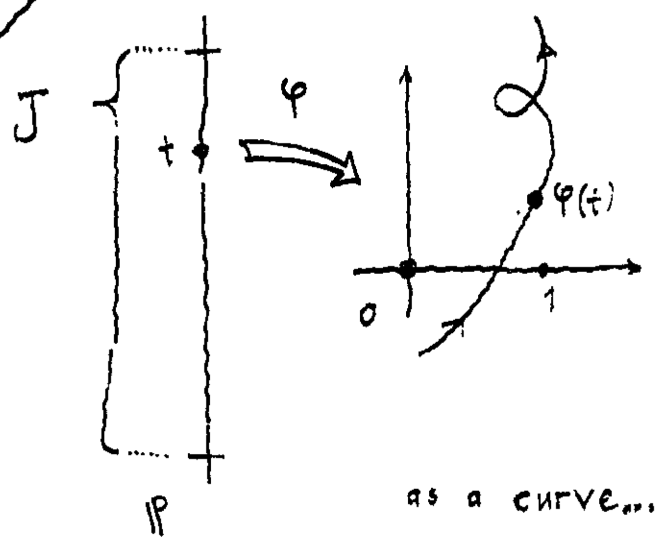
Two basic interpretations :

Either




assigning a complex number to each $t \in J$

or




as a curve...

Differentiation and integration are defined componentwise!

$$\varphi'(t) = \alpha'(t) + i\,\beta'(t)$$

$$\int^{*} \varphi(t)\,dt = \int^{*} \alpha(t)\,dt + \left( \int^{*} \beta(t)\,dt \right) i$$

$\underline{*}$ definite or indefinite!

Example: $\frac{d}{dt}\left( (t+t^2) + e^t i \right) = 1 + 2t + e^t i$

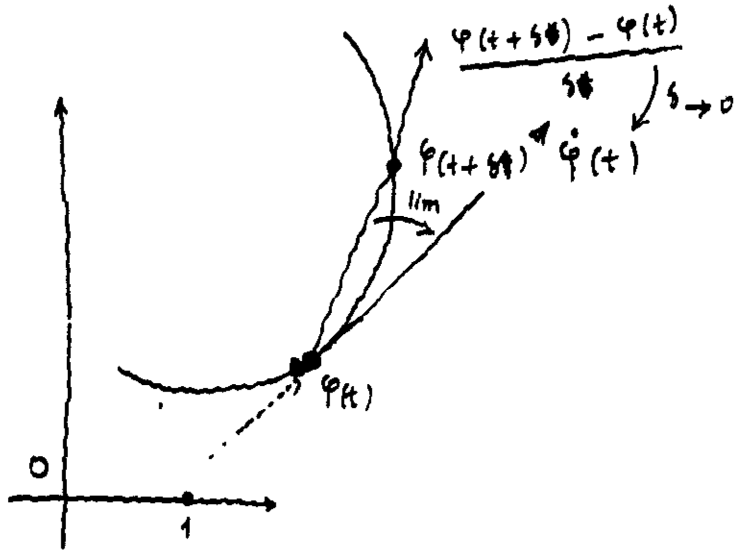
$$\int \left( (t+t^2) + e^t i \right) dt = \frac{1}{2}t^2 + \frac{1}{3}t^3 + e^t i + C$$

constant

$\in \mathbb{C}$

$$\int_0^1 \left( (t+t^2) + e^t i \right) dt = \frac{1}{2}t^2 + \frac{1}{3}t^3 + e^t i \Big|_0^1 = \frac{5}{6} + (e-1)i$$

If we interprete $\varphi : J \to \mathbb{C}$ as the ~~orbit~~ trajectory of a particle then $\dot{\varphi}(t)$ is its "instantaneous velocity" at time $t$.



If $\dot{\Psi}(t)$ is continous, then $\int \dot{\Psi}(t)\,dt = \Psi(t) + C$.

53

Properties of definite integrals :

1. $\int_a^b \varphi(t) = -\int_b^a \varphi(t)\,dt$ equivalently $\int_a^a \varphi(t)\,dt = 0$

2. $\int_a^b \varphi(t)\,dt + \int_b^c \varphi(t)\,dt = \int_a^c \varphi(t)\,dt$

3. $\int_a^b \varphi_1(t)\,dt + \int_a^b \varphi_2(t)\,dt = \int_a^b \left(\varphi_1(t) + \varphi_2(t)\right) dt$

— trivial —

4. $\left| \int_a^b \varphi(t)\,dt \right| \leqslant \int_a^b |\varphi(t)|\,dt$

Proof: Let $\int_a^b \varphi(t)\,dt = Re^{i\Theta}$ (constant). We have

$$\left| \int_a^b \varphi(t)\,dt \right| = R = \int_a^b \cancel{R}\,\varphi(t)\,e^{-i\Theta}\,dt$$

$$\overset{!}{=} \int_a^b \mathrm{Re}\left(\varphi(t)\,e^{-i\Theta}\right) dt \leqslant \int_a^b \left|\varphi(t)\,e^{-i\Theta}\right| dt$$

$$= \int_a^b |\varphi(t)|\,dt.$$

Important: If we interprete $\varphi : [a,b] \longrightarrow \mathbb{C}$ as a curve clearly $\int_a^b |\dot{\varphi}(t)|\,dt$ is its length !

Exercise: length$(\varphi) \geqslant$ dist$(\varphi(a), \varphi(b))$ .