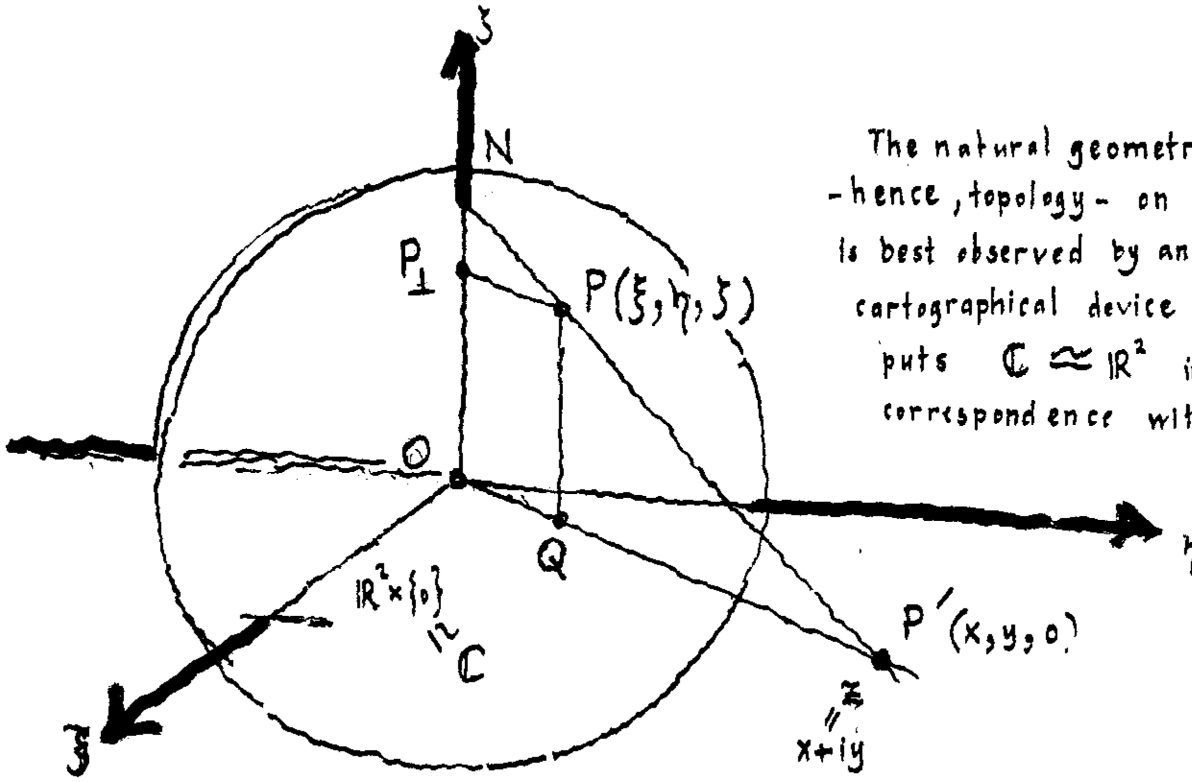
§ 20. The Riemann sphere :



The natural geometry – hence, topology – on $\mathbb{C}P^1$ is best observed by an ancient cartographical device that puts $\mathbb{C} \simeq \mathbb{R}^2$ into 1-1 correspondence with $S^2 - \{N\}$

where $\quad S^2 = \{ (\xi, \eta, \zeta) \in \mathbb{R}^3 \mid \xi^2 + \eta^2 + \zeta^2 = 1 \}$

$\qquad N = (0, 0, 1)$.

This device is the <u>stereographic projection</u> which is a map

$$\sigma : S^2 - \{N\} \subseteq \mathbb{R}^3 \longrightarrow \mathbb{C} \simeq \mathbb{R}^2 \simeq \mathbb{R}^2 \times \{0\}$$

that sends each $P \in S^2 - \{N\}$ into P' which is the point of intersection of NP with $\mathbb{C} \simeq \mathbb{R}^2 \simeq \mathbb{R}^2 \times \{0\}$.

To obtain an explicit description, it is sufficient to observe that

$$\overrightarrow{OP'} : \overrightarrow{P_\perp P} = \overrightarrow{NO} : \overrightarrow{NP_\perp}$$

hence

$$x : \xi = y : \eta = 1 : 1 - \zeta$$

and

$$\sigma((\xi, \eta, \zeta)) = \frac{\xi + i\eta}{1 - \zeta} .$$

As for the inverse of σ, it is sufficient to notice

$$x^2 + y^2 = \frac{\xi^2 + \eta^2}{(1-\zeta)^2} = \frac{1 - \zeta^2}{(1-\zeta)^2} = \frac{1+\zeta}{1-\zeta}$$

from which

$$\zeta = \frac{x^2+y^2-1}{x^2+y^2+1}, \quad \xi = \frac{2x}{x^2+y^2+1}, \quad \eta = \frac{2y}{x^2+y^2+1} .$$

Summing up:

$$\sigma^{-1}(z) = \left(\frac{2\mathrm{Re}(z)}{1+|z|^2}, \frac{2\mathrm{Im}(z)}{1+|z|^2}, \frac{|z|^2-1}{|z|^2+1} \right) .$$

$S^2 - \{N\}$ corresponds to $\mathbb{C}$ & N represents "the point at infinity"

§21. The Cauchy-Riemann Equations:

Consider $f : \Omega \underset{op}{\subseteq} \mathbb{C} \longrightarrow \mathbb{C}$ with $f(z) = u(x,y) + v(x,y)i$ and $z = x + iy$, generically.

Theorem: If f is differentiable at $a = \alpha + i\beta \in \Omega$ then u, v have partial derivatives at (α, β) and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at (α, β).

Proof:

$$f'(a) = \lim_{\substack{\delta \in \mathbb{R} \\ \delta \to 0}} \frac{f(a+\delta) - f(a)}{\delta} = \lim_{\substack{\delta \to 0 \\ \delta \in \mathbb{R}}} \left[\frac{u(\alpha+\delta, \beta) - u(\alpha, \beta)}{\delta} + \frac{v(\alpha+\delta, \beta) - v(\alpha, \beta)}{\delta} i \right]$$

$$\overset{(*)}{=} \left. \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i \right|_{(\alpha,\beta)}$$

On the other hand

$$f'(a) = \lim_{\substack{\delta \to 0 \\ \delta \in \mathbb{R}}} \frac{f(a+i\delta) - f(a)}{i\delta}$$

$$= \lim_{\substack{\delta \to 0 \\ \delta \in \mathbb{R}}} \left[\frac{u(\alpha, \beta+\delta) - u(\alpha,\beta)}{i\delta} + \frac{v(\alpha, \beta+\delta) - v(\alpha,\beta)}{i\delta} i \right]$$

$$\overset{(**)}{=} -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \Bigg|_{(\alpha,\beta)}$$

Compare $(*)$ and $(**)$!

Example : $$\underset{f(z)}{z^3} = x^3 - 3xy^2 + (3x^2y - y^3)i$$

Notice that the Ca~~u~~uchy - Riemann equations are necessary for differentiability but by no means ~~necessary~~ sufficient !

Example : $f : \mathbb{C} \longrightarrow \mathbb{C}$

defined by

$$f(z) = \begin{cases} \dfrac{\bar{z}^3}{|z|^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 . \end{cases}$$

As $u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \longrightarrow \left.\dfrac{\partial u}{\partial x}\right|_{(0,0)} = 1 \quad \& \quad \dfrac{\partial u}{\partial y} = 0$

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and

$v(x,y) = \begin{cases} \dfrac{-3x^2y + y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \longrightarrow \dfrac{\partial v}{\partial x} = 0 \quad \& \quad \dfrac{\partial v}{\partial y} = +1$

f is seen to satisfy the Cauchy-Riemann equations at $(0,0)$. ~~but~~ However f is <u>not</u> differentiable at $z = 0$ since

$\dfrac{f(\overset{\neq 0}{\zeta}) - f(0)}{\zeta} = \dfrac{\bar{\zeta}^2}{\zeta^2}$ which can attain any complex value ~~which can be any ,~~ of unit modulus.

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PROBLEMS (7)

1. Prove that $f : \mathbb{C} \to \mathbb{C}$, defined by

$$f(z) = \bar{z}$$

is nowhere differentiable .

2. Prove that $f : \mathbb{C} \to \mathbb{C}$, defined by

$$f(z) = x^3 + (1-y)^3 i$$

is differentiable only at $z = i$. Evaluate $f'(i)$.

3. Consider $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{for} \quad z \neq 0 \\ 0 & \text{for} \quad z = 0 \end{cases}$$

(A) Prove that f is a continous function,

(B) Prove that the real and imaginary parts of f satisfy the Cauchy-Riemann equations at $z = 0$.

(C) Prove that $f(z)$ is not differentiable at $z = 0$.

4. Consider $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \dfrac{x^3 y(y - xi)}{x^6 + y^2} & \text{for} \quad z \neq 0 \\ 0 & \text{for} \quad z = 0 \end{cases}$$

Prove that

$$\frac{f(z) - f(0)}{z} \to 0$$

as $z \to 0$ along any line through $z = 0$ but $f(z)$ is not differentiable at $z = 0$.

§22. A partial converse of the Cauchy-Riemann equations

Theorem: If the real & imaginary parts of $f : \Omega \xrightarrow{\subseteq_{\text{op}} \mathbb{C}} \mathbb{C}$ have partial derivatives in a neighbourhood of $a \in \Omega$ which satisfy the Cauchy-Riemann equations at a and are continuous at $a \in \Omega$, then f is differentiable at $a \in \Omega$.

Proof:

$$\frac{f(\overset{\alpha+i\beta}{a}+\overset{\xi+i\eta}{\delta}) - f(a)}{\delta} = \frac{u(\alpha+\xi,\beta+\eta) - u(\alpha,\beta) + \big(v(\alpha+\xi,\beta+\eta) - v(\alpha,\beta)\big)i}{\xi+i\eta}$$

by the mean value theorem!

$$= \frac{\xi \left.\frac{\partial u}{\partial x}\right|_{\alpha+s\xi,\beta+\eta} + u(\alpha,\beta+\eta) - u(\alpha,\beta) + \left(\xi \left.\frac{\partial v}{\partial x}\right|_{\alpha+s'\xi,\beta+\eta} + v(\alpha,\beta+\eta) - v(\alpha,\beta)\right)i}{\xi+i\eta}$$

where $s, s' \in (0,1)$

$$= \frac{\xi \left.\frac{\partial u}{\partial x}\right|_{\alpha+s\xi,\beta+\eta} + \eta \left.\frac{\partial u}{\partial y}\right|_{\alpha,\beta+t\eta} + \left(\xi \left.\frac{\partial v}{\partial x}\right|_{\alpha+s'\xi,\beta+\eta} + \eta \left.\frac{\partial v}{\partial y}\right|_{\alpha,\beta+t'\eta}\right)i}{\xi+i\eta}$$

where $t, t' \in (0,1)$.

$$= \frac{\xi\left(\frac{\partial u}{\partial x}\Big|_{\alpha,\beta} + \Delta_1\right) + \eta\left(\frac{\partial u}{\partial y}\Big|_{\alpha,\beta} + \Delta_2\right) + \xi\left(\frac{\partial v}{\partial x}\Big|_{\alpha,\beta} + \Delta_3\right)i + \eta\left(\frac{\partial v}{\partial y}\Big|_{\alpha,\beta} + \Delta_4\right)i}{\xi + i\eta}$$

where $\Delta_k \to 0$ as $\overset{\delta}{\overset{\shortparallel}{\xi + i\eta}} \to 0$

$k = 1, 2, 3, 4$!

$$= \frac{\xi \frac{\partial u}{\partial x} - \eta \frac{\partial v}{\partial x} + \xi \frac{\partial v}{\partial x} i + \eta \frac{\partial u}{\partial x} i}{\xi + i\eta}\Bigg|_{(\alpha,\beta)} + \frac{\xi}{\xi + i\eta}(\Delta_1 + i\Delta_3) + \frac{\eta}{\xi + i\eta}(\Delta_2 + i\Delta_4)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Bigg|_{(\alpha,\beta)} + \underbrace{\frac{\xi}{\xi + i\eta}(\Delta_1 + i\Delta_3) + \frac{\eta}{\xi + i\eta}(\Delta_2 + i\Delta_4)}_{\to\, 0 \text{ as } \delta = \xi + i\eta \to 0}$$

since $\left|\frac{\xi}{\xi + i\eta}\right|, \left|\frac{\eta}{\xi + i\eta}\right| \leq 1$.