§19. Linear fractional transformations :

Start with a special case : Consider $f : \mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$

where $f(z) = \frac{1}{z}$. Clearly $f \circ f = Id$.

(i.e. f is "involutive" !)

note that $f(\mathbb{C} - \{0\}) \subseteq \mathbb{C} - \{0\}$.

Under f the set of points satisfying

$$az\bar{z} + \bar{b}z + b\bar{z} + c = 0$$

is transformed into the set of points satisfying

$a, c \in \mathbb{R}$
$b \in \mathbb{C}$
& $|b|^2 - ac > 0$.

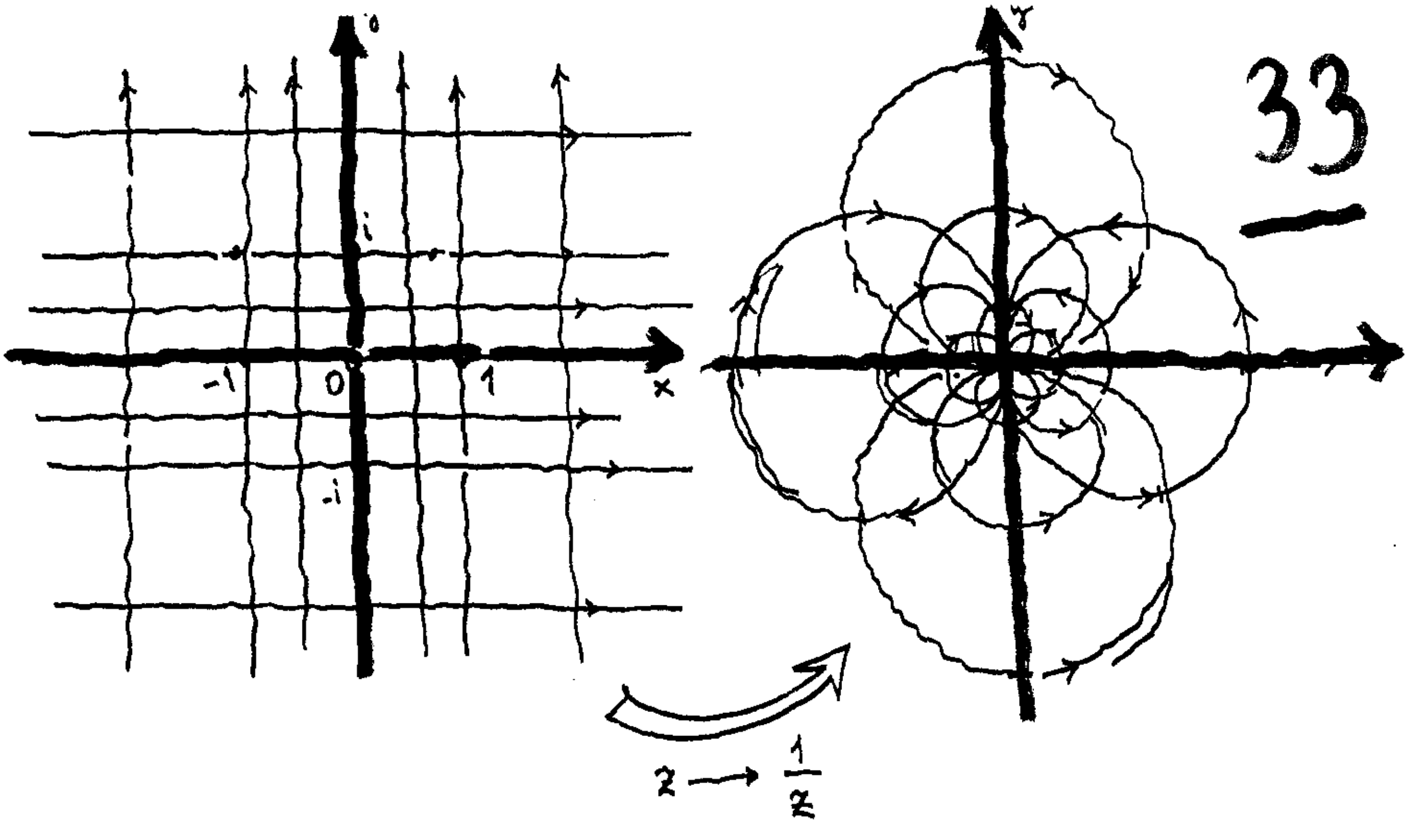
$$cz\bar{z} + \bar{b}\bar{z} + bz + a = 0.$$

Taking $a = 0$, it is seen that (1) a line not through 0 (i.e. $c \neq 0$) turns into a circle through 0 (2) a line through 0 (i.e. $c = 0$) turns into a line (into itself!) through 0.

Taking $a \neq 0$, it is seen that (3) a circle through 0 (i.e. $c = 0$) turns into a line through 0, (4) a circle not through 0 (i.e. $c \neq 0$) turns into ~~a line~~ a circle not through 0.

* Start to think of a line as a "generalised circle" *

a circle through "infinity"



More generally,

A linear fractional transformation is a map

$$f : \mathbb{C} - \left\{ \frac{-d}{c} \right\} \longrightarrow \mathbb{C}$$

where $f(z) = \dfrac{az + b}{cz + d}$ for $\forall\ z \in \mathbb{C} - \left\{ \frac{-d}{c} \right\}$

~~w.~~ with $c \neq 0$ (for $c = 0$ the above formula describes a similarity transformation, already considered...)

and $ad - bc \neq 0$ (for $ad - bc = 0$ reduces $\frac{az+b}{cz+d}$ to the constant b/d !)

f is clearly differentiable and

~~It can be check~~

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \quad \forall\ z \neq -\frac{d}{c} .$$

f never takes the value $\frac{a}{c}$ and in fact f is a bijection from $\mathbb{C} - \left\{-\frac{d}{c}\right\}$ to $\mathbb{C} - \left\{\frac{a}{c}\right\}$ with

$$f^{-1}(z) = \frac{-dz+b}{cw-a} .$$

Since

$$f(z) = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} = \frac{a}{c} + \frac{\frac{bc-ad}{c}}{cz+d}$$

f is seen to be a concatenation of Euclidean transformations and the special case $z \longrightarrow \frac{1}{z}$ we have started with :

$z \longrightarrow cz$ dilation about origin by factor $|c|$ followed by rotation about origin through ~~Arg(z).~~ Arg(c) .

$z \longrightarrow z + d$ translation by d

$z \longrightarrow \frac{1}{z}$

$z \longrightarrow \overbrace{\frac{bc-ad}{c}}^{=\varepsilon} z$ dilation about origin by factor $|\varepsilon|$ followed rotation about origin through ~~Arg(z)~~ Arg(ε) .

$z \longrightarrow z + \frac{a}{c}$ translation by $\frac{a}{c}$.

Great simplifications ensue in connection with linear fractional transformations if ~~the complex plane~~ $\mathbb{C}$ is augmented by an ideal point ∞ "the point at infinity" which is assumed to lie at an infinitely large distance from all points of $\mathbb{C}$. The resulting set $\mathbb{C} \cup \{\infty\}$ will be called (for our purposes!) "the extended complex plane and be denoted by $\mathbb{C}P^1$.

A linear fractional transformation $\overset{-\frac{d}{c}}{\overset{*}{z}} \longrightarrow \dfrac{az+b}{cz+d}$ can be envisaged, to great advantage, as a map $\varphi : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ defined by

$c \neq 0$

$$\varphi(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } z \neq -\frac{d}{c} \text{ or } \infty \\[2ex] \infty & \text{if } z = -\frac{d}{c} \\[2ex] a/c & \text{if } z = \infty \end{cases}$$

$c = 0$

($d \neq 0$ since $ad - bc \neq 0$)

$$\varphi(z) = \begin{cases} \dfrac{az+b}{d} & \text{if } z \neq \infty \\[2ex] \infty & \text{if } z = \infty . \end{cases}$$

The statements involving ∞ in the above definition of φ have their intuitive origins in the facts that

$$\left.\begin{aligned}&\lim_{z \to -d/c} \frac{az+b}{cz+d} = \infty \\ &\lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c}\end{aligned}\right\} \text{ for } c \neq 0$$

and

$$\lim_{z \to \infty} \frac{az+b}{d} = \infty \qquad \text{for } c = 0 .$$

A very complete intuitive picture is achieved by imagining lines in $\mathbb{C}$ as "generalised circles" in $\mathbb{CP}^1$ through ∞.

PROBLEMS (6)

1. Remember the van der Mond determinant :

$$det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (a-b)(b-c)(c-a) \ .$$

Let $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ stand for the extended complex plane.

(A) Prove that a linear fractional transformation that leaves three <u>distinct</u> points of $\mathbb{C}P^1$ is the identity map.

(B) Given distinct $a, b, c \in \mathbb{C}P^1$ prove that there exists a linear fractional transformation that maps a, b, c into $0, 1, \infty \in \mathbb{C}P^1$ respectively.

(C) Given distinct $a, b, c \in \mathbb{C}P^1$ and distinct $a', b', c' \in \mathbb{C}P^1$ prove that there exists a <u>unique</u> linear fractional transformation that maps a, b, c into a', b', c' respectively.

(D) Construct a linear fractional transformation that maps the open unit disc $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ onto the upper half plane $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

(E) Construct a linear fractional transformation that maps the open disc $\Delta = \{z \in \mathbb{C} \mid |z-5| < 5\}$ onto the half plane $H = \{z \in \mathbb{C} \mid \mathrm{Re}(z) + \mathrm{Im}(z) > 7\}$.

(F) Construct a linear fractional transformation that maps the crescent

$$\Gamma = \{z \in \mathbb{C} \mid |z+1|^2 > 2 \ , \ |z-1|^2 < 2\}$$

onto the " first quadrant " $H = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0 \ , \ \mathrm{Im}(z) > 0\}$.