§15. Functions of a complex variable:

are functions of the form $f : \Omega \subseteq \mathbb{C} \longrightarrow \overset{\text{any set.}}{X}$. For our purposes $\Omega \underset{op}{\subseteq} \mathbb{C}$, $X = \mathbb{C}$ always.

Example: $f : \mathbb{C} \longrightarrow \mathbb{C}$

defined by $f(z) = |z| + \mathrm{Re}(z)\, i$

(putting $z = x + iy$ generically) $= \underbrace{(x^2+y^2)^{1/2}}_{u} + \underbrace{x}_{v}\, i$

(or putting $z = r\,E(\theta)$ generically) $= \overbrace{r} + \overbrace{(r\cos\theta)}\, i$

A function $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ will be frequently written in the form

$$f(z) = u(x,y) + v(x,y)\, i$$

using generically $z = x + iy$, with $u, v : \Omega \subseteq \mathbb{C} \simeq \mathbb{R}^2 \longrightarrow \mathbb{R}$,

or

$$f(z) = U(r,\theta) + V(r,\theta)\, i$$

where $x = r\cos\theta$, $y = r\sin\theta$ etc.

Example: $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = z^3$

$$\therefore \quad f(z) = (x+iy)^3 = x^3 + 3x^2y\,i - 3xy^2 - iy^3$$

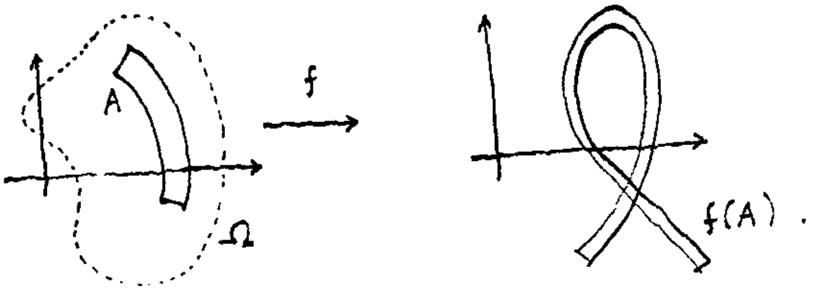
$$f(z) = \underbrace{x^3 - 3xy^2}_{u(x,y)} + (3x^2y - y^3)\, i$$

$$\cancel{= r^3\cos(3\theta) + r^3\sin(3\theta)\, i}$$

or

$$f(z) = \underbrace{r^3\cos(3\theta)}_{U(r,\theta)} + \underbrace{r^3\sin(3\theta)}_{V(r,\theta)}\, i$$

§16. Functions of complex variables as mappings :



Example: $f : \mathbb{C} \longrightarrow \mathbb{C}$, $\boxed{f(z) = z^2} = x^2 - y^2 + (2xy)\,i$

$$\left(\frac{Y}{2a}\right)^2 = a^2 - X$$

X

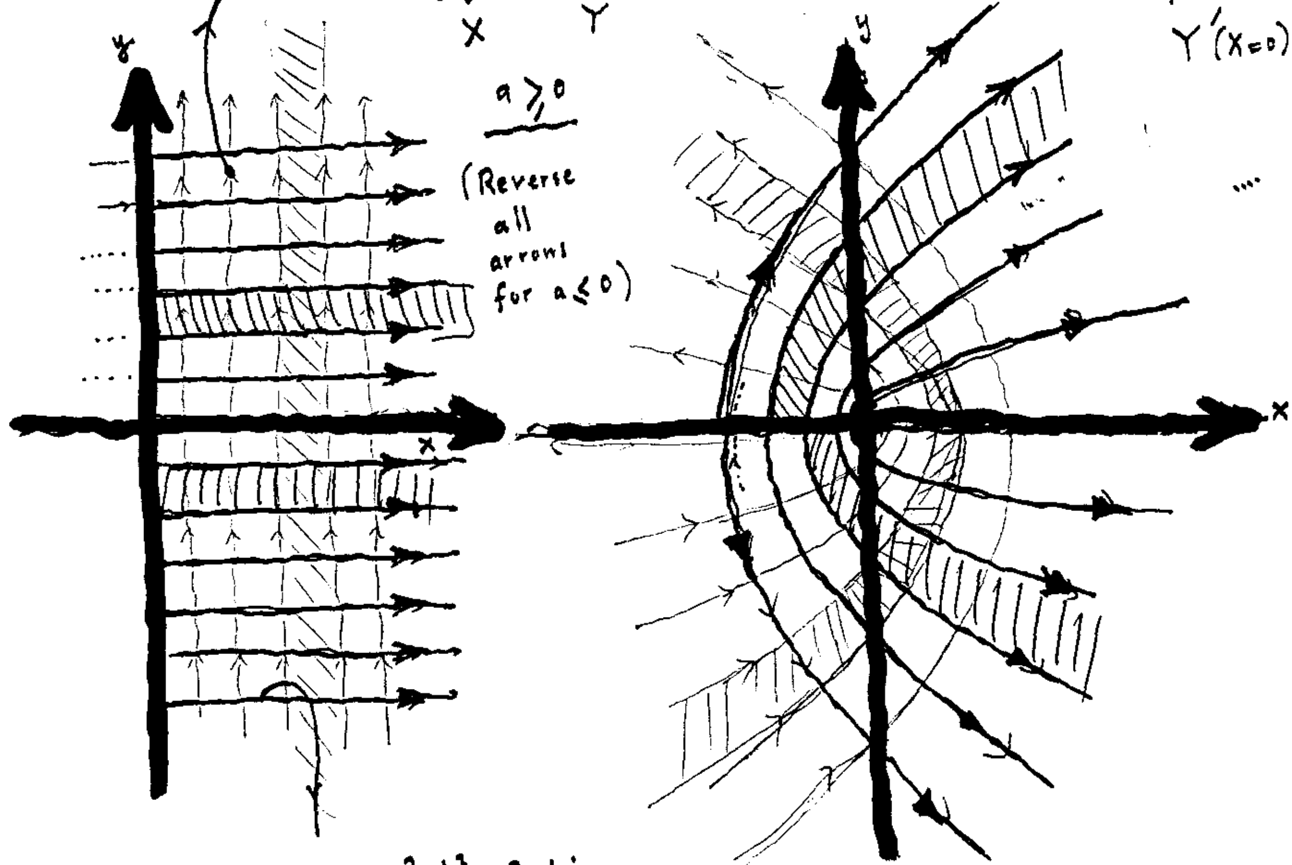
$Y(0) = 2a^2$

$X(0) = a^2$

Y

$Y'(X=0) = -1$

$a + ti \longrightarrow \underbrace{a^2 - t^2}_{X} + \underbrace{2at}_{Y} i$

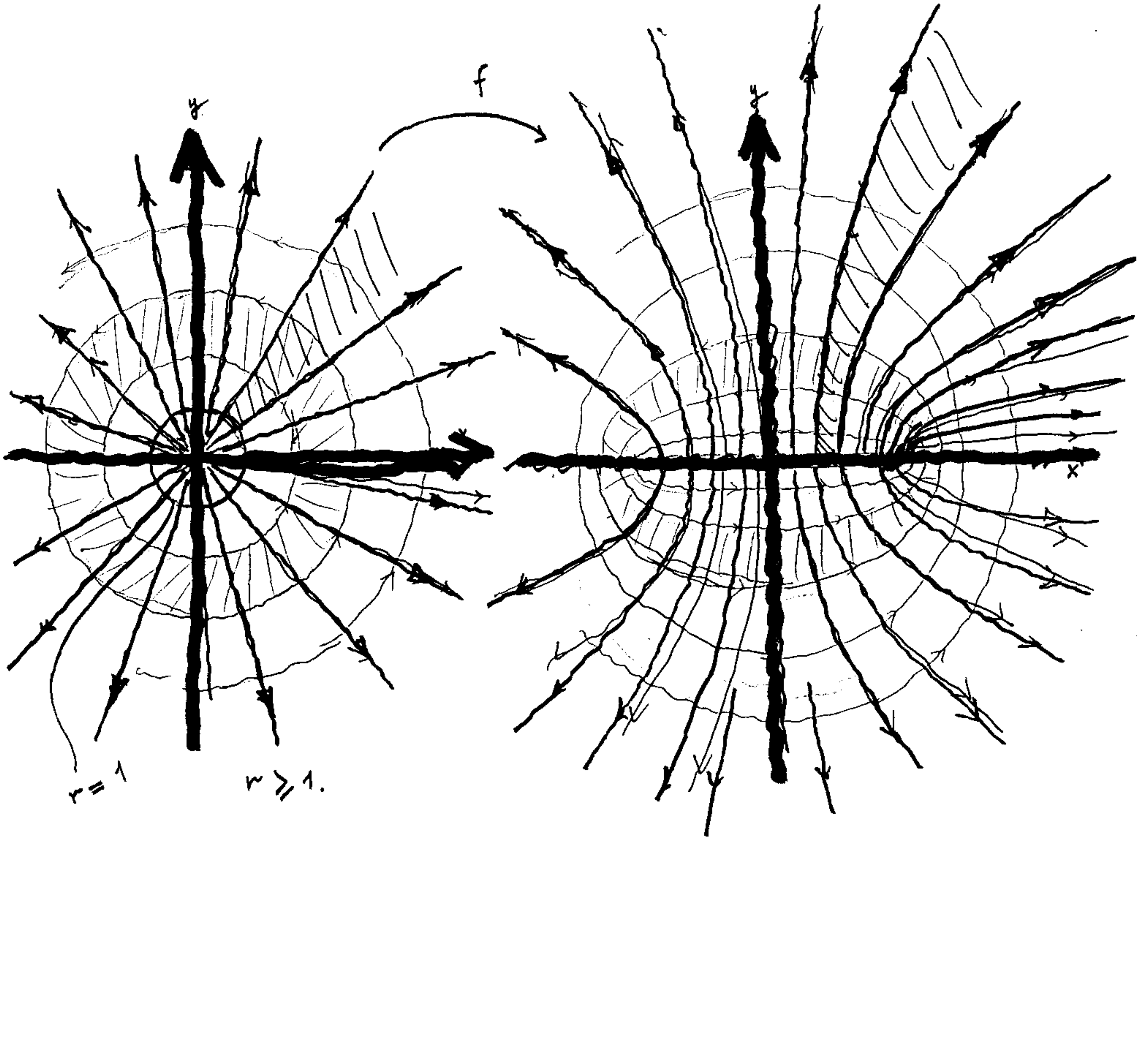


$s + bi \longrightarrow s^2 - b^2 + 2sbi$

Example: $f : \mathbb{C} - \{0\} \longrightarrow \mathbb{C}$

$$\boxed{f(z) = z + \frac{1}{z}}$$

$$f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$



§17. Differentiability with respect to a complex variable:

Definition: $f : \Omega \underset{op}{\subseteq} \mathbb{C} \longrightarrow \mathbb{C}$ is said to be *differentiable at* $a \in \Omega$, if $\lim\limits_{z \to a} \dfrac{f(z) - f(a)}{z - a}$ exists. This limit is called the *derivative of* f at $z = a$. Notation $f'(a)$, $\dfrac{df}{dz}(a)$ etc.

f is said to be *differentiable on* Ω if f is differentiable at every $a \in \Omega$. If f is differentiable on Ω, $f' : \Omega \to \mathbb{C}$ is the function that assigns to each $\cancel{\zeta}$ $z \in \Omega$ the derivative of f at z.

Example: $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$ *differentiable on* $\mathbb{C}$.

$$\lim_{0 \neq \zeta \to 0} \frac{f(z+\zeta) - f(z)}{\zeta} = \lim_{\zeta \to 0} \frac{(z+\zeta)^2 - z^2}{\zeta} = \lim_{\zeta \to 0} (2z + \zeta) = 2z.$$

Thus $(z^2)' = 2z$. Quite generally $(z^n)' = n z^{n-1}$ for all $n \in \mathbb{Z}$, $n \geqslant 0$. (In fact we may allow $n < \infty$ if we take $\operatorname{dom} f = \mathbb{C} - \{0\}$.)

Example: $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = |z|^2$ is differentiable only at $z = 0$. Indeed

$$\frac{f(\zeta) - f(0)}{\zeta} = \frac{|\zeta|^2}{\zeta} = \bar{\zeta} \longrightarrow 0 \quad \text{as } \zeta \to 0.$$

But for $z \neq 0$

$$\frac{f(z+\zeta) - f(z)}{\zeta} = \frac{|z+\zeta|^2 - |z|^2}{\zeta} = \frac{(z+\zeta)(\bar{z}+\bar{\zeta}) - z\bar{z}}{\zeta}$$

$$= \bar{z} + \frac{\bar{\zeta}}{\zeta} z + \bar{\zeta}$$

No limit since $\bar{\zeta}/\zeta$ can be any number of unit modulus.

Example: $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = |z|$ is nowhere differentiable.

$$\frac{f(0+\zeta) - f(0)}{\zeta} = \frac{|\zeta|}{\zeta} \quad \text{no limit!}$$

$z \neq 0$
$$\frac{f(z+\zeta) - f(z)}{\zeta} = \frac{|z+\zeta| - |z|}{\zeta} = \frac{|z+\zeta|^2 - |z|^2}{\zeta\left(|z+\zeta| + |z|\right)}$$

as above!
$$= \frac{\bar{z} + \overbrace{\frac{\bar{\zeta}}{\zeta}}^{\text{no limit}} + \bar{\zeta}}{|z+\zeta| + |z|}$$

$\longrightarrow 2|z|$

It can be routinely checked that differentiation obeys the <u>usual rules</u>:

~~Given~~ If $f, g : \Omega \overset{\subseteq\,\mathbb{C}}{\longrightarrow} \mathbb{C}$ are differentiable at $a \in \Omega$ then so is $f+g$, $f \cdot g$, $\frac{f}{g}$ (provided $g(a) \neq 0$) and

$$(f+g)'(a) = f'(a) + g'(a)$$

"The Leibniz Rule"

$$(fg)'(a) = f'(a)\,g(a) + f(a)\,g'(a)$$

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\,g(a) - f(a)\,g'(a)}{g(a)^2}.$$

"The Chain Rule"

If $f : \Omega \subseteq_\circ \mathbb{C} \longrightarrow \mathbb{C}$ is differentiable at $a \in \Omega$, $f(a) \in \Lambda = \mathrm{dom}(g)$ and g is differentiable at $f(a) \in \Lambda$ then so is $g \circ f$ at a and

$$(g \circ f)'(a) = g'(f(a))\,f'(a).$$

Of course, if f is differentiable at $a \in \mathrm{dom}(f)$, then f is continuous at a.

§18. <u>Conformality</u>:

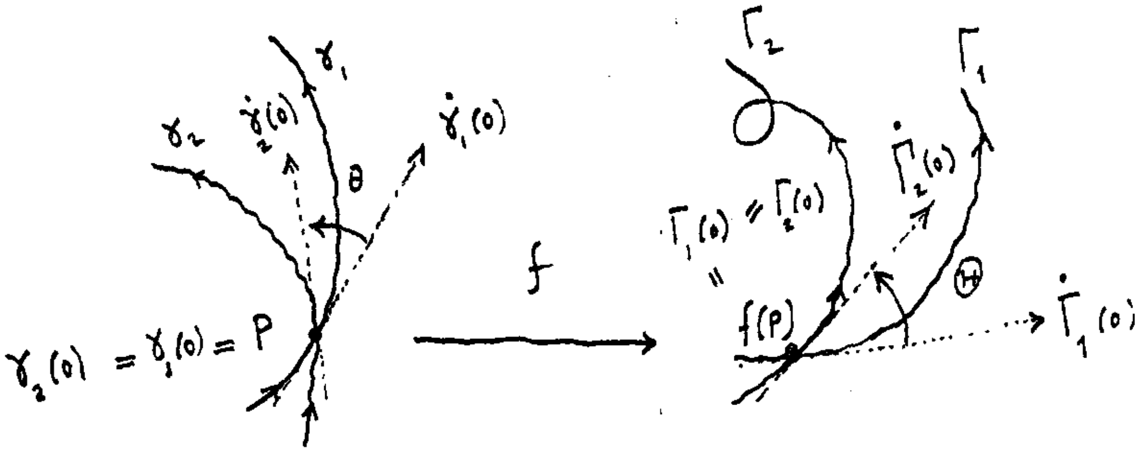
A differentiable function $f : \Omega \longrightarrow \mathbb{C}$ is <u>conformal</u> at points where f' does <u>not</u> vanish, in the sense that it preserves angles between curves and preserves orientation.

Given differentiable curves $\gamma_1, \gamma_2 : J \xrightarrow{\subseteq \mathbb{R}} \mathbb{C}$ intersecting in $P = \gamma_1(0) = \gamma_2(0)$

J a neighbourhood of 0, with $\dot{\gamma}_1(0) \neq 0$ & $\dot{\gamma}_2(0) \neq 0$,

$\cancel{\measuredangle_P \gamma_1}$

we understand the angle between γ_1 and γ_2 at P to be the angle between the vectors $\dot{\gamma}_1(0)$, $\dot{\gamma}_2(0)$. Consider $f : \Omega \longrightarrow \mathbb{C}$ with $P \in \Omega$ and with f differentiable at P. Let $\Gamma_1 = f \circ \gamma_1$, $\Gamma_2 = f \circ \gamma_2$.



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Let θ be the angle between γ_1 & γ_2 at P and Θ be the angle between Γ_1 & Γ_2 at $f(P)$.

elements of $\mathbb{R}/2\pi\mathbb{Z}$

Clearly

$$\Theta = \operatorname{Arg}\left(\frac{\dot{\Gamma}_2(0)}{\dot{\Gamma}_1(0)}\right) = \operatorname{Arg}\left(\frac{\overset{\gamma_2(0)}{\overset{\parallel}{f'(P)\,\dot{\gamma}_2(0)}}}{\underset{\gamma_1(0)}{\underset{\parallel}{f'(P)\,\dot{\gamma}_1(0)}}}\right)$$

$$= \operatorname{Arg}\left(\frac{\dot{\gamma}_2(0)}{\dot{\gamma}_1(0)}\right) = \theta \quad ,$$

provided $f'(P) \neq 0$.