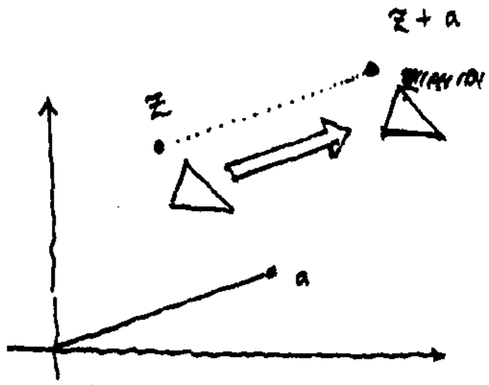
§11. Euclidean transformations :

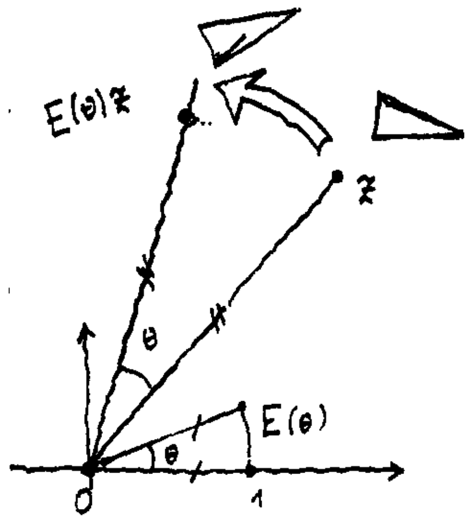
Addition of a fixed complex number results in a "translation" :

$$z \longrightarrow z + a$$



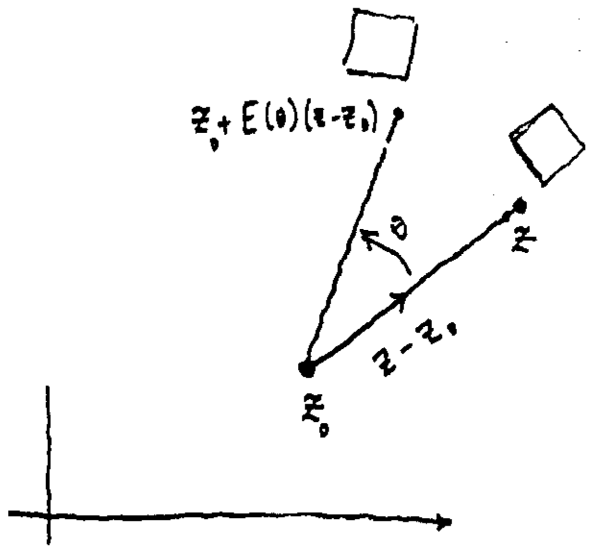
Multiplication by $E(\theta)$ ~~is~~ results in a rotation about $0 \in \mathbb{C}$ through angle θ.

$$z \longrightarrow E(\theta)\, z$$

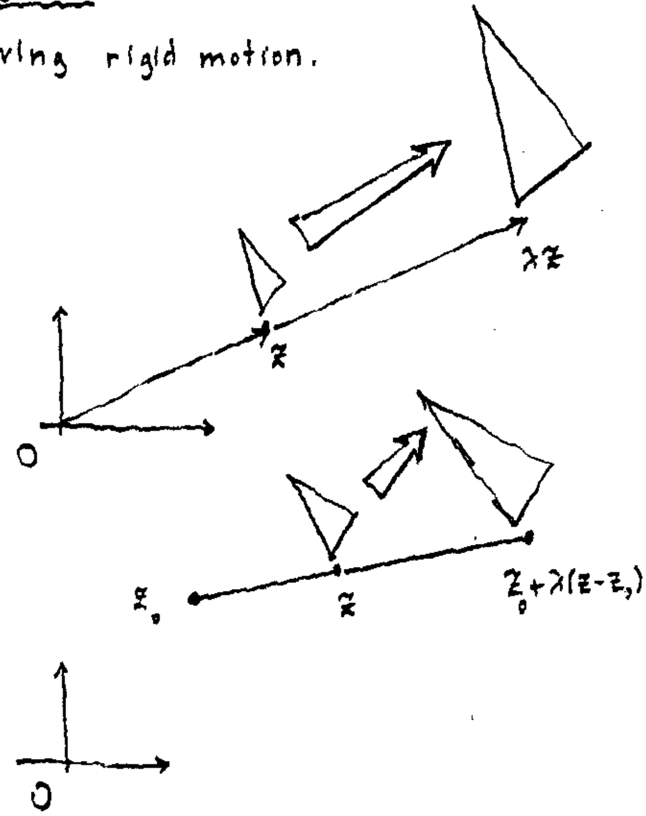


Rotation about $z_0 \in \mathbb{C}$ through angle θ

$$z \longrightarrow z_0 + E(\theta)\,(z - z_0)$$



~~Any~~ A composition of translation of translations and rotations is called a (n orientation preserving) rigid motion. Two geometric objects are called (directly) congruent if the one results from the other by an orientation preserving rigid motion.



Multiplication by $\lambda \in \mathbb{R} - \{0\}$ is a dilation about ~~the origin~~ $0 \in \mathbb{C}$ by factor λ:

$$z \longrightarrow \lambda z$$

Dilation about $z_0 \in \mathbb{C}$ by factor $\lambda \in \mathbb{R} - \{0\}$:

$$z \longrightarrow z_0 + \lambda(z - z_0)$$

A composition of (o.p.) rigid motions and dilations is called a (n orientation preserving) similarity transformation. Two geometric objects are called (directly) similar if the one is obtained from the other by an o.p. similarity transformation.

PROBLEMS (4)

1. Prove that triangles with vertices a, b, c and a', b', c' are (directly) similar iff

$$det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a' & b' & c' \end{bmatrix} = 0 \ .$$

2. (a) Prove that $a, b, c \in \mathbb{C}$ constitute the vertices of an equilateral triangle iff

$$det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ b & c & a \end{bmatrix} = 0 \ .$$

(b) Prove that the roots of the equation

$$z^3 + Az^2 + Bz + C = 0$$

constitute the vertices of an equilateral triangle iff $A^2 = 3B$.

(c) Prove that $0, a, b \in \mathbb{C}$ constitute the vertices of an equilateral triangle iff $|a|^2 = |b|^2 = 2\mathrm{Re}(a\bar{b})$.

§ 12. Topology of $\mathbb{C}$:

The distance function $d(z,w) = \text{dist}(z,w) = |z-w|$ is easily seen to satisfy the axioms for a metric function. The topology of $\mathbb{C}$ is the metric topology induced by d. As such $\mathbb{C}$ is indistunguishable from $\mathbb{R}^2$. So we freely discourse about limits, continuity and about sets that are open, closed, bounded, compact, connected and the concepts of boundar~~ies~~y, closure~~s~~, interior.

Important Facts :

1. A continuous real valued function is bounded from above & below and attains its least upper & greatest lower bounds.

2. A continous real valued function with a connected domain, that takes negative and positive values attains 0.

PROBLEMS (5)

1. In each of the following cases decide if the set of complex numbers z which satisfy the given relation is open closed, bounded, compact, connected :

(a) $|z-i| \leq 1$

(b) $|z-2| > |z-1|$

(c) $\left|\dfrac{z-1}{z+1}\right| = 1$

(d) $|z^2-1| < 1$

(e) $|z^2| = \mathrm{Im}(z)$

(f) $\mathrm{Re}(\dfrac{z}{1+i}) = 0$

(g) $\mathrm{Re}((1-i)\bar{z}) = 0$

§13. The fundamental theorem of algebra:

Theorem: Every non-constant polynomial with complex coefficients has a root in $\mathbb{C}$.

Proof: (Elementary proof due to E. Landau ≈ 1910). Consider non-constant $p(X) \in \mathbb{C}[X]$

1° $p(z) \longrightarrow \infty$ as $z \longrightarrow \infty$. $\therefore$ $\exists\, z_0 \in \mathbb{C}$ such that

$$\inf_{z \in \mathbb{C}} |p(z)| = |p(z_0)| \overset{\neq 0}{>0}$$

2° Assume w.l.o.g. that $z_0 = 0$: Replace, if necessary, $p(z)$ by $p(z + z_0)$.

3° Assume w.l.o.g. that $p(0) = 1$: Replace, if necessary $p(z)$ by $\left(p(0)\right)^{-1} p(z)$

4° $\therefore$ For some $m \geq 1$ & $n \geq m$

$$p(z) = 1 + \overset{\neq 0}{a_m} z^m + a_{m+1} z^{m+1} + \dots + a_n z^n$$

and we may assume w.l.o.g. that $a_m = -1$: Replace, if necessary $p(z)$ by $p(\omega z)$ where $\omega^m = -1/a_m$.

5° For $t \in [0,1]$

$$|p(t)| = \left|1 - t^m + \sum_{k=m+1}^{n} a_k t^k\right| \leq 1 - t^m + \left(\sum_{k=m+1}^{n} |a_k|\right) t^{m+1}$$

But: RHS < 1 if $t < \left(\sum_{k=m+1}^{n} |a_k|\right)^{-1}$.

§ 14. History:

1°/ Since long it ~~was~~ is known that some quadratic equations are not ~~being~~ "solvable" over $\mathbb{R}$. In the early 16. century the expression "not solvable" quadratic started to be replaced by "quadratic with imaginary roots".

2°/ The use of imaginary quantities grew in importance as identities like

$$4 = \sqrt[3]{2+11\sqrt{-1}} + \sqrt[3]{2-11\sqrt{-1}}$$

(Rafael Bombelli ~ 1560)

started to occur more frequenty, in late 16th century. The above particular identity results when one solves $x^3 - 15x - 4 = 0$ the method of Cardano and notes that 4 is a root.

3°/ Leonhardt Euler introduces notation "i". Notices that $E(\theta)$ (1777) is $e^{i\theta}$, formally by writing

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \cdots$$

$$= \underbrace{\left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots\right)}_{\sin\theta.}$$

4°/ The Interpretation of $\mathbb{C}$ as a model of the Euclidean plane is due to Caspar Wessel Danish (in 1799)

Rediscovered by Jean-Robert Argand in 1806

by Carl Friedrich Gauss in 1831

5°/ C.F. Gauss introduces the terminology "Komplexe Zahl" ~ 1830

6°/ Abstract definition of complex numbers as ordered pairs of real numbers :

William Rowan Hamilton in 1837

& higher dimensional analogues of $\mathbb{C}$:

"Classical number systems"

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \underbrace{\mathbb{C}}_{\dim = 2} \subseteq \underbrace{\mathbb{H}}_{\dim = 4} \subseteq \underbrace{\mathbb{O}}_{\dim = 8}$$

"quaternions", Hamilton 1843

"octonions"

John T. Graves 1843

Arthur Cayley 1850

7°/ Georg Friedrich Bernhard Riemann :

the "Riemann Sphere" ~ 1860.