§1. The naive approach to complex numbers:

Proceed with computations just as in $\mathbb{R}$, the only novelty being the presence of an "imaginary number" i with $\boxed{i^2 = -1}$:

$$\begin{aligned}(5+2i)(3+4i) &= 5\cdot 3 + 5\cdot 4i + 2i\cdot 3 + 2i\cdot 4i \\ &= 15 + 20i + 6i - 8 \\ &= 7 + 26i\end{aligned}$$

More generally with $a, b, c, d \in \mathbb{R}$

$$(a+bi)(c+di) = ac - bd + (ad+bc)i \, .$$

§2. The field of complex numbers: "$\mathbb{C}$"

$\mathbb{C}$ is $\mathbb{R}^2$ (with its usual structure as vector space) with

"addition" $\rightarrow$ the usual

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$$

"multiplication"

$$\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix} .$$

It can be routinely checked that $\mathbb{C}$ is a commutative ring with multiplicative identity $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and additive identity $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Not routine is the presence of multiplicative inverses of non-zero elements. Indeed, for $\begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (a^2+b^2)^{-1} a \\ -(a^2+b^2)^{-1} b \end{bmatrix}.$$

$\mathbb{C}$ is a field.

Notation : Put $"i" = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $"1" = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and write $\begin{bmatrix} a \\ b \end{bmatrix} = a . "1" + b . "i"$ as $a + bi$

§3. Alternative Definitions of $\mathbb{C}$:

$\sim a + bi$ with $"1" \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $"i" = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\mathbb{C} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

with the usual matrix addition & multiplication

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$$\mathbb{C} = \frac{\mathbb{R}[x]}{\langle x^2+1 \rangle}$$

$\mathbb{R}[x]$ ~ polynomials with real coefficients in indeterminate x.

$\langle x^2+1 \rangle$ the ideal generated by x^2+1. It consists of polynomials in $\mathbb{R}[x]$ that are divisible by x^2+1.

"declare $x^2 = -1$" so that $a + bx + \langle x^2+1 \rangle$ $\sim a + bi$

§4. Real numbers as a subfield:

In whichever way that we introduce $\mathbb{C}$, we understand $\mathbb{R}$ as a subfield thereof by identifying $a \in \mathbb{R}$ with $a + 0i \in \mathbb{C}$.

Terminology & Notation:

$z = x + yi$ generically

with $x \in \mathbb{R}$ "real part" $\mathrm{Re}(z)$

$y \in \mathbb{R}$ "imaginary part" $\mathrm{Im}(z)$

Clearly $z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$.

The **conjugate** $\bar{z}$ of $z \in \mathbb{C}$ is defined by

$$\bar{z} = \mathrm{Re}(z) - i\,\mathrm{Im}(z) .$$

That is,

$$z = x + yi \longleftrightarrow \bar{z} = x - yi .$$

Observe:

$$\left.\begin{aligned} z + \bar{z} &= 2\mathrm{Re}(z) \\ z - \bar{z} &= 2i\,\mathrm{Im}(z) \end{aligned}\right\} \text{hence} \quad \begin{aligned} \mathrm{Re}(\bar{z}) &= \mathrm{Re}(z) \\ \mathrm{Im}(\bar{z}) &= -\mathrm{Im}(z) \end{aligned}$$

$$\bar{\bar{z}} = z$$

$$\overline{zw} = \bar{z}\,\bar{w} .$$

We say "z is real" if $\mathrm{Im}(z) = 0$,
"z is purely imaginary" if $\mathrm{Re}(z) = 0$.

For example, for any $z \in \mathbb{C}$
the number $\bar{z} z$ is ~~purely~~ real.