MATH 353 - COMPLEX CALCULUS

FIRST MIDTERM

FAMILY NAME	OTHER NAMES	GRADE

13th October 2005. Duration : 90 minutes. Three questions : 15 + 15 , 10 + 15 + 15 , 15 + 15

Solutions

1. (A) Prove that $|z|^2 + a^2 = |z+a|^2 - 2\mathrm{Re}(az)$ for any $a \in \mathbb{R}$, and $z \in \mathbb{C}$.
(B) Prove that $|z|^2 + 2\mathrm{Re}(bz) = |z+\bar{b}|^2 - |b|^2$ for any $b, z \in \mathbb{C}$.

Let $z = x + yi$:

A) $|z|^2 + a^2 = x^2 + y^2 + a^2 = (x+a)^2 + y^2 - 2ax$

$= |z+a|^2 - 2\mathrm{Re}(az)$

Put $b = \alpha + \beta i$:

B) $|z|^2 + 2\mathrm{Re}(bz) = x^2 + y^2 + 2(\alpha x - \beta y)$

$= (x+\alpha)^2 + (y-\beta)^2 - (\alpha^2 + \beta^2)$

$= |z+\bar{b}|^2 - |b|^2$

2. (A) If $\alpha \in \mathbb{C}$ is a root of a polynomial g with real coefficients, prove that $\bar{\alpha}$ is also a root of g.

(B) Find $a, b \in \mathbb{R}$ such that $f(z) = z^3 + az + b$ admits $1 + 2i$ as a root.

(C) With these values of a, b substituted in f, find the remaining roots of f.

(A) Given polynomial $g(z) = A_0 + A_1 z + \cdots + A_n z^n$ with root $\alpha \in \mathbb{C}$, where $A_0, A_1, \ldots, A_n \in \mathbb{R}$,

$$g(\bar{\alpha}) = A_0 + A_1\bar{\alpha} + \cdots + A_n \bar{\alpha}^n = \overline{(A_0 + A_1\alpha + \cdots + A_n \alpha^n)} = \overline{g(\alpha)} = 0.$$

$\therefore$ $\bar{\alpha}$ is a root of g, too.

(B) $$f(1+2i) = (1+2i)^3 + a(1+2i) + b = 0$$

$$= -11 + a + b + (-2 + 2a)i = 0$$

$\therefore$ $a = 1$ $\quad b = 10$.

(C) Since $1+2i$, $1-2i$ are roots of f, the third root $c \in \mathbb{C}$ must satisfy

$$(1+2i) + (1-2i) + c = 0.$$

Hence $c = -2$.

3. (A) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a function defined by $f(z) = (z^2 - 1)\mathrm{Im}(z)$. Find the derivative of f at all the points at which f is differentiable.

(B) Let $\Omega \subseteq_{op} \mathbb{C}$ be a connected set and $g : \mathbb{C} \longrightarrow \mathbb{C}$ be a differentiable function such that $|g(z)|$ is constant on Ω. Prove that g is constant on Ω.

Let $z = x + yi$ generically.

(A) $f = \underbrace{(x^2 - y^2 - 1)y}_{u} + \underbrace{2xy^2}_{v}\, i$

Cauchy-Riemann Eqs are satisfied at points where f is differentiable

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \longrightarrow xy = 0 \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \longrightarrow x^2 - y^2 = 1 \end{array}\right\} \begin{array}{l} f \text{ is } \underline{\text{nowhere differentiable}} \\ \text{except at } z = \pm 1. \end{array}$$

$$\frac{f(1 + \overset{\xi + i\eta}{\overset{\|}{\delta}}) - f(1)}{\delta} = \frac{((1+\delta)^2 - 1)\eta - 0}{\delta} = (2 + \delta)\eta \xrightarrow{\text{as } \delta \to 0} 0$$

$\therefore$ f is differentiable at $z = +1$ and $f'(1) = 0$.
Similarly at $z = -1$ and $f'(-1) = 0$.

(B) Let $f = u + vi$ as usual. Differentiate $u^2 + v^2 =$ constant w.r.to x and y to obtain

$$\left.\begin{array}{r} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x} = 0 \\[2ex] u\dfrac{\partial u}{\partial y} + v\dfrac{\partial v}{\partial y} = -v\dfrac{\partial u}{\partial x} + u\dfrac{\partial v}{\partial x} = 0 \end{array}\right\} \begin{bmatrix} u & v \\ -v & u \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{bmatrix} = 0 \longrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$$

&

similarly

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0$$

whenever $u^2 + v^2 \neq 0$.

$\therefore$ $f' \equiv 0$.

$\therefore$ Domain of f being connected

$f \equiv$ constant.