Advanced Calculus

M A T H 2 5 1

**MAKE-UP EXAMINATION**

(*Duration : 120 mins.*)

27th January 1997

[ 5 + 5 + 15 ] , [ 25 ] , [ 20 ] , [ 15 + 15 ]

**1.**

(a) Give the definition of a connected set.

(b) Given $K \subseteq \mathbb{R}^n$ with the property that for any $a \in K, \quad b \in K$ there exists a continuous map $\varphi : [\ 0,1\ ] \longrightarrow K$ such that $\varphi(0) = a$ , $\varphi(1) = b$ , prove that $K$ is a connected set.

(c) Prove that the set

$$K = \{(x,y) \in \mathbb{R}^2 \mid y > x^2\} \cup \{(0,0)\}$$

is connected.

**2.**

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{for} \quad (x,y) \neq (0,0) \\ 0 & \text{for} \quad (x,y) = (0,0) \end{cases}$$

Prove that $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$ exist but they are not equal .

**3.**

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function with the property that for any open $U \subseteq \mathbb{R}$ , $f(U)$ is open, too. Prove that the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = |f(x)|$ for $x \in \mathbb{R}$ cannot attain a maximum.

P L E A S E   T U R N   O V E R   !

**4.**

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n : \mathcal{D} \subseteq \mathbb{R}^p \longrightarrow \mathbb{R}$ which is monotone decreasing in the sense that $f_m(x) \geq f_n(x)$ for any $x \in \mathcal{D}$ and $m, n \in \mathbb{N}$ with $m \leq n$ .

(a) If $\lim_{n\longrightarrow\infty} f_n(c) = 0$ for $c \in \mathcal{D}$ , prove that given $\varepsilon > 0$ , there exists $N_0 \in \mathbb{N}$ and a neighbourhood $V$ of $c$ such that $0 \leq f_n(x) < \varepsilon$ for $n \geq N_0$ and $x \in V \cap \mathcal{D}$ .

(b) Let $\mathcal{D}$ be compact, $\lim_{n\longrightarrow\infty} f_n(x) = 0$ for each $x \in \mathcal{D}$ . Prove that $f_n$ converges to 0 uniformly on $\mathcal{D}$ .