

Advanced Calculus

M A T H 2 5 1

SECOND MIDTERM

(Duration : 100 mins.)

15th December 1996

$$[10 + 10 + (10 + 5)] , [(4 + 4) + (4 + 4) + (4 + 4) + (4 + 4 + 4) + 4] , [(5 + 5) + 15]$$

1.

Consider the sequence $(x_n)_{n \in \mathbb{N}}$ defined recursively by

$$x_1 = 421 \quad , \quad x_{n+1} = \frac{x_n^2 + 3}{2x_n}$$

- (a) Prove that $x_n > \sqrt{3}$ for all $n \in \mathbb{N}$. (*Hint* : Consider $(x_n - \sqrt{3})^2 > 0$.)
- (b) Prove that $x_{n+1} < x_n$ for all $n \in \mathbb{N}$.
- (c) Is $(x_n)_{n \in \mathbb{N}}$ convergent ? If it is, compute its limit.

2.

Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x + 2 & \text{for } x \leq 0 \\ -x + 3 & \text{for } x > 0 \end{cases}$$

- (a) Find open sets $U, V \subseteq \mathbb{R}$ such that $f^{-1}(U)$ is open, $f^{-1}(V)$ is not open, (Write out these latter sets explicitly !)

(b) Find connected sets $A, B \subseteq \mathbb{R}$ such that $f(A)$ is connected, $f(B)$ is not connected. (Write out these latter sets explicitly !)

(c) Find compact sets $K, L \subseteq \mathbb{R}$ such that $f(K)$ is compact, $f(L)$ is not compact. (Write out these latter sets explicitly !)

(d) Find convergent sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}, (z_n)_{n \in \mathbb{N}}$ in \mathbb{R} with respective limits x, y, z such that $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(x)$, $(f(y_n))_{n \in \mathbb{N}}$ is convergent but does not converge to $f(y)$, $(f(z_n))_{n \in \mathbb{N}}$ is divergent. (Write out these latter sequences explicitly !)

(e) Is f continuous ? Be brief !

P L E A S E T U R N O V E R !

3.

(a) Let $G : [a, b] \rightarrow \mathbb{R}$ be differentiable on $[a, b]$. If G attains a minimum (over $[a, b]$) at a , prove that $G'(a) \geq 0$. What happens if G attains a minimum (over $[a, b]$) at b ?

(b) Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable on $[a, b]$ with

$$f'(a) = K < L = f'(b).$$

Given $M \in (K, L)$, consider the function $F : [a, b] \rightarrow \mathbb{R}$ defined by

$$F(x) = f(x) - M(x - a)$$

and prove that there exists $c \in (a, b)$ such that $f'(c) = M$.