Small Skew Fields

Cédric Milliet — Université Claude Bernard, Lyon 1

Antalya Modnet Conference, November 2006

1 - Definitions

Definition — A *skew field*, or simply a *field*, is a ring $\langle D, 0, 1, +, \times \rangle$ with $\times$ not necessarily commutative, and any non zero element invertible.

1 - Definitions

Definition — A *skew field*, or simply a *field*, is a ring $\langle D, 0, 1, +, \times\rangle$ with $\times$ not necessarily commutative, and any non zero element invertible.

Definition — A theory $\mathcal{T}$ is *small* if $S_n(\mathcal{T})$ is countable for each $n < \omega$. A model $\mathcal{M}$ is small if its theory $\mathcal{T}(\mathcal{M})$ is small.

2 - Motivations

Small theories

Vaught's conjecture

2 - Motivations

Small theories

Vaught's conjecture

Skew fields

Finite fields

2 - Motivations

Small theories

Vaught's conjecture

Skew fields

Finite fields
$\omega-$stable fields

2 - Motivations

Small theories

Vaught's conjecture

Skew fields

Finite fields
$\omega-$stable fields
Superstable fields

2 - Motivations

Small theories

Vaught's conjecture

Skew fields

Finite fields
$\omega-$stable fields
Superstable fields
Supersimple fields

2 - Motivations

Small theories

Vaught's conjecture

Skew fields

Finite fields
ω−stable fields
Superstable fields
Supersimple fields

Theorem — An infinite small commutative field is algebraically closed.

2 - Motivations

Small theories

Vaught's conjecture

Skew fields

Finite fields
$\omega-$stable fields
Superstable fields
Supersimple fields

Theorem — An infinite small commutative field is algebraically closed.

Question — Is a small skew field commutative ?

2 - Motivations

Small theories

Vaught's conjecture

Skew fields

Finite fields
$\omega-$stable fields
Superstable fields
Supersimple fields

Theorem — An infinite small commutative field is algebraically closed.

Question — Is a small skew field commutative ?

Theorem — A small skew field of positive characteristic is commutative.

3 - Properties

Properties — Let $\mathcal{M}$ be a small $\mathcal{L}-$strucutre.

1. Let $\overline{a}$ be finite in M. Then the $\{\mathcal{L} \cup \overline{a}\}-$structure $\mathcal{M}$ is small.
2. Let X be definable in M^n. Then X is small.
3. Let X be definable in M^k and E a definable equivalence relation on M^k. Then X/E is small.
4. If $\varphi : G \to G$ is an injective definable morphism of a definable group $G \subset \mathcal{M}$, then it is surjective.

Definition — Let $\mathcal{M}$ be a structure, $A \subset \mathcal{M}$ and D a definable set. We define $CB_A(D)$ by

1. $CB_A(D) \geq 0$ if $D \neq \emptyset$
2. $CB_A(D) \geq \alpha+1$ if there is an infinite family $(D_i)_{i\in\omega}$ of $A-$definable disjoint sets such that $D_i \subset D$ and $CB_A(D_i) \geq \alpha$ for all $i \in \omega$.
3. $CB_A(D) \geq \beta$ for a limit ordinal β if $CB_A(D) \geq \alpha$ for all $\alpha < \beta$.

Definition — The Cantor-Bendixson degre $dCB_A(D)$ of D over A is the greatest integer d such that D can be divided into d disjoint $A-$definable sets with same rank over A.

$\omega-stable\ groups$

if $G \lneq_{def} H$, then

$$(MR(G), dM(G)) \lneq (MR(H), dM(H))$$

$\omega-stable\ groups$	$Small\ groups$
if $G \lneq_{def} H$, then $(MR(G), dM(G)) \lneq (MR(H), dM(H))$	if $G \cap dcl(\overline{h}) \lneq H \cap dcl(\overline{h})$, then $(CB_{\overline{h}}(G), dCB_{\overline{h}}(G)) \lneq (CB_{\overline{h}}(H), dCB_{\overline{h}}(H))$

$\omega-stable\ groups$	$Small\ groups$
if $G \lneq_{def} H$, then $(MR(G), dM(G)) \lneq (MR(H), dM(H))$	if $G \cap dcl(\overline{h}) \lneq H \cap dcl(\overline{h})$, then $(CB_{\overline{h}}(G), dCB_{\overline{h}}(G)) \lneq (CB_{\overline{h}}(H), dCB_{\overline{h}}(H))$

Proposition *(Descending chain Condition)* — Let G be a small group and $G' < G$ a finitely generated definable closure in G. There is no infinite descending chain of subgroups of the form

$$G_0 \cap G' \gneq G_1 \cap G' \gneq G_2 \cap G' \gneq ...$$

where the G_i are subgroups of G definable over G'.

ω−*stable groups*	*Small groups*
if $G \lneqq_{def} H$, then $(MR(G), dM(G)) \lneqq (MR(H), dM(H))$	if $G \cap dcl(\overline{h}) \lneqq H \cap dcl(\overline{h})$, then $(CB_{\overline{h}}(G), dCB_{\overline{h}}(G)) \lneqq (CB_{\overline{h}}(H), dCB_{\overline{h}}(H))$

Proposition *(Descending chain Condition)* — Let G be a small group and $G' < G$ a finitely generated definable closure in G. There is no infinite descending chain of subgroups of the form

$$G_0 \cap G' \gneqq G_1 \cap G' \gneqq G_2 \cap G' \gneqq ...$$

where the G_i are subgroups of G definable over G'.

(Counter) Examples

1. $\mathbb{Q}$

ω−stable groups	*Small groups*
if $G \lneq_{def} H$, then $(MR(G), dM(G)) \lneq (MR(H), dM(H))$	if $G \cap dcl(\overline{h}) \lneq H \cap dcl(\overline{h})$, then $(CB_{\overline{h}}(G), dCB_{\overline{h}}(G)) \lneq (CB_{\overline{h}}(H), dCB_{\overline{h}}(H))$

Proposition *(Descending chain Condition)* — Let G be a small group and $G' < G$ a finitely generated definable closure in G. There is no infinite descending chain of subgroups of the form

$$G_0 \cap G' \gneq G_1 \cap G' \gneq G_2 \cap G' \gneq ...$$

where the G_i are subgroups of G definable over G'.

(Counter) Examples

1. $\mathbb{Q}$
2. $\mathbb{H}_{\mathbb{Q}}$

Lemma 1 — Let D be a small field, and a in D. Then, for all $n > 0$, $C_D(a) = C_D(a^n)$.

Lemma 1 — Let D be a small field, and a in D. Then, for all $n > 0$, $C_D(a) = C_D(a^n)$.

Lemma 2 — Let D be a small field, $x \notin Z(D)$, and $\varphi : x \mapsto x^a - x$. Then $1 \notin im(\varphi)$.

Lemma 1 — Let D be a small field, and a in D. Then, for all $n > 0$, $C_D(a) = C_D(a^n)$.

Lemma 2 — Let D be a small field, $x \notin Z(D)$, and $\varphi : x \mapsto x^a - x$. Then $1 \notin im(\varphi)$.

Theorem — A small field of positive characteristic is commutative.