G-compactness and groups

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- T – complete theory in language L

- $\mathfrak{C} \models T$ monster model (saturated)

- $M \models T$ is small if $M \prec \mathfrak{C}$ and $|M| = |T|$

- The group of Lascar strong automorphisms:
  $\mathrm{Autf_L}(\mathfrak{C}) = \left\langle \mathrm{Aut}(\mathfrak{C}/M) \colon M \text{ small model } \right\rangle$

- The finest bounded equivalence relation i.e. equality of Lascar strong types:
  $E_L(a,b) \iff \exists f \in \mathrm{Autf_L}(\mathfrak{C}),\ a = f(b)$

- The finest bounded $\emptyset$-type definable equivalence relation i.e. equality of Kim-Pillay strong types E_{KP}:
  $E_{KP}(a,b) \iff \exists f \in \mathrm{Autf_{KP}}(\mathfrak{C}),\ a = f(b)$

- The intersection of all finite definable equivalence relations i.e. equality of Shelah strong types E_{Sh}:
  $E_{Sh}(a,b) \iff (\exists f \in \mathrm{Autf}_{\mathrm{Sh}}(\mathfrak{C}))\ a = f(b)$

- formula $\varphi(x,y) \in L$ is *thick* iff
  $\forall (a_i)_{i<\omega} \subseteq \mathfrak{C}\ \exists\ i < j < \omega,\ \varphi(a_i, a_j)$

- Θ the type of all thick formulas:

$$\Theta(x,y) = \bigwedge_{\varphi \text{ thick}} \varphi(x,y)$$

- $E_L(x,y)$ is the transitive closure of $\Theta(x,y)$

- *Lascar group* of T:
  $\mathrm{Gal}_{\mathrm{L}}(T) = \mathrm{Aut}(\mathfrak{C}) / \mathrm{Autf}_{\mathrm{L}}(\mathfrak{C})$

$\mathsf{Gal}_{\mathsf{L}}(T)$ is compact topological group:
(not necessary Hausdorff)

N, M small models,

$$\begin{array}{ccc} \mathsf{Aut}(\mathfrak{C}) & \xrightarrow{j=\nu\circ\mu} & \mathsf{Gal}_{\mathsf{L}}(T) = \mathsf{Aut}(\mathfrak{C})/\,\mathsf{Autf}_{\mathsf{L}}(\mathfrak{C}) \\ \mu\downarrow & \nearrow_{\nu} & \\ S_M(N) & & \end{array}$$

$$\begin{array}{ccc} f & \xmapsto{j=\nu\circ\mu} & f\cdot\mathsf{Autf}_{\mathsf{L}}(\mathfrak{C}) \\ \mu\downarrow & \nearrow_{\nu} & \\ \mathsf{tp}(f[M]/N) & & \end{array}$$

$$S_M(N) = \{\mathsf{tp}(M'/N) \colon \mathsf{tp}(M') = \mathsf{tp}(M)\}$$

$\mathsf{Gal}_{\mathsf{L}}(T)$ carries an induced topology from ν:

$$X \underset{\text{closed}}{\subseteq} \mathsf{Gal}_{\mathsf{L}}(T) \overset{\text{definition}}{\iff} \nu^{-1}[X] \underset{\text{closed}}{\subseteq} S_M(N)$$

T is G-compact iff this topolgy is Hausdorff iff E_L is type definable iff $E_L = E_{KP}$

Definition 1. For $G < \mathrm{Aut}(\mathfrak{C})$ define an equivalence relation E_G:

$$E_G(a,b) \iff \exists f \in G,\ a = f(b)$$

E. g. $E_L = E_{\mathrm{Autf_L}(\mathfrak{C})}$, $E_{KP} = E_{\mathrm{Autf_{KP}}(\mathfrak{C})}$

Theorem 2 (Lascar, 2001)**.** *If $\overline{E_L}$ is the topological closure of E_L in $S_2(\emptyset)$, then*

$$E_{KP} = \Theta \circ \overline{E_L}$$

If $j : \mathrm{Aut}(\mathfrak{C}) \longrightarrow \mathrm{Gal_L}(T)$ a quotient map, then $\mathrm{Autf_{KP}}(\mathfrak{C}) = j^{-1}[\ \overline{\mathrm{id}}\ ]$.

Theorem 3. *For* $\mathrm{Autf_L}(\mathfrak{C}) \lhd\ G < \mathrm{Aut}(\mathfrak{C})$ *let* $\overline{\overline{G}} = j^{-1}[\ \overline{j[G])}\ ]$. *Then*
(i) $E_{\overline{\overline{G}}}$ is the finest bounded type-definable over any small model equivalence relation extending E_G
(ii) if additionally $G \lhd \mathrm{Aut}(\mathfrak{C})$, then

$$E_{\overline{\overline{G}}} = \Theta \circ \overline{E_G}.$$

Proof. The crucial point is:

Theorem 4 (Ziegler)**.** *(Weak openness of ν) For $p \in S_M(N)$ define its Θ-neighbourhood*

$$[p]_\Theta = \{q \in S_M(N)\colon p(x) \cup q(y) \cup \Theta(x,y) \text{ is cons.}\}$$

Then $\forall p \in S_M(N),\ U \subseteq S_M(N)$

$$[p]_\Theta \subseteq \mathrm{int}(U) \ \Rightarrow\ \nu(p) \in \mathrm{int}(\nu[U]).$$

□

Recall that E_{Sh} is the intersection of all $\emptyset$-definable finite equivalence relations. Let QC(1) be the intersection of all open subgroup of $\mathrm{Gal}_\mathrm{L}(T)$ (i.e. Quasi Component). Then

$$\mathrm{Autf}_{\mathrm{Sh}}(\mathfrak{C}) = j^{-1}[\mathrm{QC}(1)].$$

Remark 5. For $H < \mathrm{Aut}(\mathfrak{C})$ let $\mathrm{QC}(H)$ be the intersection of all open subgroup containing H. Then $E_{j^{-1}[\mathrm{QC}(H)]}$ is the intersection of all $\emptyset$-definable finite equivalence relations extending $E_{j^{-1}[H]}$.

An example

Let G be a group, and consider the two-sorted structure

$$\mathcal{G} = (G, X, \cdot),$$

where $\cdot\colon G \times X \to X$ is a regular action of G on the X i.e. X is affine copy of G. Fix an arbitrary point $x_0 \in X$ and assume that $\mathcal{G}$ is saturated. Then $X = G \cdot x_0$, so we can define homomorphic embedding:

$$\mathrm{Aut}(G) \ni f \mapsto \overline{f} \in \mathrm{Aut}(\mathcal{G}),$$
$$G \ni g \mapsto \overline{g} \in \mathrm{Aut}(\mathcal{G}),$$

$\overline{f}_{|G} = f$ and $\overline{f}(h \cdot x_0) = f(h) \cdot x_0$
$\overline{g}_{|G} = \mathrm{id}$ and $\overline{g}(h \cdot x_0) = (hg^{-1}) \cdot x_0$, for $h \in G$

Special case: if G is a compact Lie group then in the paper of Ziegler it is proved that

$$\mathrm{Gal}_{\mathrm{L}}(\mathbb{R}, G, X, \cdot) = G.$$

What do E_L, E_{KP}, Θ, $\mathrm{Gal_L}(\mathcal{G})$,... look like? We will anwser this in general.

Definition 6. $X_\Theta = \{a \cdot b^{-1} \colon \Theta(a,b),\ a,b \in G\}$,

$$X_L = \{a \cdot b^{-1} \colon E_L(a,b),\ a,b \in G\},$$
$$G_L = \langle X_L \rangle = \langle X_\Theta \rangle < G.$$

For relation E on G, let $X_E = \{a \cdot b^{-1} \colon E(a,b)\}$, $G_E = \langle X_E \rangle$. If e.g. $E(x,y) \Leftrightarrow \exists z,\ x = y^z$, then $X_E = \{[x,y] \colon x,y \in G\},\ G_E = [G,G]$.

Proposition 7. *(i)* $\mathrm{Aut}(\mathcal{G}) = G \rtimes \mathrm{Aut}(G)$, $\Big(\overline{\mathrm{Aut}(G)}$ *acts on* $\overline{G}$*:* $\ \overline{g}^{\overline{f}} = \overline{f}\overline{g}\overline{f^{-1}} = \overline{f(g)}\Big)$

(ii) $\mathrm{Autf_L}(\mathcal{G}) = G_L \rtimes \mathrm{Autf_L}(G)$, $\Big(F \in \mathrm{Aut}(\mathcal{G}/\mathcal{G}_0) \Rightarrow F = \overline{f(h)h^{-1}} \circ \overline{f}$, *where* $f = F_{|G} \in \mathrm{Aut}(G/G_0)\Big)$

(iii) $\mathrm{Gal_L}(Th(\mathcal{G})) = G/G_L \rtimes \mathrm{Gal_L}(Th(G))$, $\Big($*subspace topology on* G/G_L*:*

$$X \underset{\textit{closed}}{\subseteq} G/G_L \quad \Leftrightarrow \quad j^{-1}[X] \underset{\textit{type def.}}{\subseteq} G\Big)$$

(iv) G_L *is the smallest* $\emptyset$*-invariant, subgroup of* G *of bounded index*

(v) for $x, y \in X$*,*

$$\Theta(x,y) \;\Leftrightarrow\; \exists g \in X_\Theta \; y = g \cdot x,$$
$$E_L(x,y) \;\Leftrightarrow\; \exists g \in G_L \; y = g \cdot x,$$
$$E_{KP}(x,y) \;\Leftrightarrow\; \exists g \in G^{00}_\emptyset \; y = g \cdot x,$$

$\Big($$G^{00}_\emptyset$ *is the smallest* $\emptyset$*-type-definable subgroup of* G *of bounded index*$\Big)$

(vi) $Th(\mathcal{G})$ *is* G*-compact* $\Leftrightarrow$ $Th(G)$ *is* G*-compact and* G_L *is type definable.*

If G_L is not type definable, then in $\mathcal{G} = (G, X, \cdot)$ the structure on the X is not G-compact and Θ-diameters of all Lascar strong type of elements X is ∞, so it would be a new example of non-G-compact theory.

How construct a group G in wich $G_L = \langle X_\Theta \rangle$ is not type definable?

Theorem 8 (Newelski, 2003)**.** *Let G be group, $X \subset G$ type def. subset and $H = \langle X \rangle$. Then*

H *is type def.* $\Leftrightarrow \exists n < \omega \; H = (X \cdot X^{-1})^n$

If in a group G, for every $n < \omega$, the set $(X_{\ominus})^n$ is not a group, then G_L is not type definable.

Let $n < \omega$. Assume that there exist an [abelian] group G_n, for wich the set $(X_{\ominus})^n$ is not a group. Consider free [abelian] group with ω generators F_ω. Then we can find a generic subset $P_n \subseteq F_\omega$ (i.e. finitely many translations cover F_ω) such that in the structure

$$(F_\omega, \cdot, P_n)$$

the set $(X_{\ominus})^n$ is also not a group. Thus in the product $G = ((F_\omega)^\omega, \cdot, P'_n)_{n<\omega}$ group G_L is not type definable. Hence it suffices to looking at the generic subsets of free [abelian] group F_ω.

There is an example of group G with $X_{\ominus} = G \setminus \{g\}$, but we do not know any example in wich e.g. $(X_{\ominus})^2$ is not a group.