

MATH 471 - Fall 2008

Hyperbolic Geometry

Schedule : Mon. 8:40-10:30 (M-103)

Wed. 9:40-10:30 (M-103)

Course outline :

Parallel postulate and the need for non-Euclidean geometry, models of the hyperbolic plane, Möbius group, classification of Möbius transformations, classical geometric notions such as length, distance, isometry, parallelism, convexity, area, trigonometry in the hyperbolic plane, group acting on the hyperbolic plane, fundamental domains.

Prerequisite: MATH 252.

Textbook : J.W. Anderson, *Hyperbolic Geometry*, Springer UMS (1999).

Grading :

Midterm 1	Oct. 25, 2008	30 %
Midterm 2	Nov. 22, 2008	30 %
Final	Jan. 13, 2009 at 16:30	40 %

MATH 471

Problem Set 1

Consider $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ as a surface of revolution with the metric given by the restriction of the standard metric on \mathbb{R}^3 .

1. Write coordinate charts explicitly and compute the matrix $[g_{ij}]$ representing the first fundamental form. Compute the Christoffel symbols.
2. Write the geodesic equations and show that at $(0, 0, 1) \in S^2$, the geodesics are precisely the meridians.
3. Verify that $\mathbf{O}(3)$, the group of 3x3 real orthogonal matrices, acts on S^2 transitively as isometries. Deduce that at any $p \in S^2$, the geodesics are the big circles through p .

Determine the **diameter** of S^2 .

4. Show that the stereographic projection $\Phi : \mathbb{R}^2 \rightarrow S^2$ is conformal, that is Φ preserves the angles between curves.
5. A curve on S^2 is called a *loxodrome* if it makes the same fixed angle θ with all latitudes. Using Problem 4, express this property in terms of a differential equation in \mathbb{R}^2 . Determine the loxodromes for $\theta = \pi/2$ and $\theta = \pi/4$.
6. For each loxodrome γ you found in Problem 5, find a transformation $T \in \text{Mob}^+$ which leaves γ invariant. Verify that T is loxodromic !
7. Suppose that $T \in \text{Mob}^+$ is non-elliptic. Show that if we have

$$\lim_{n \rightarrow \infty} T^{m(n)}(z) = z_0$$

for some z and for some sequence $\{m(n) \in \mathbb{Z} : \lim_{n \rightarrow \infty} m(n) = \infty\}$, then z_0 is a fixed point of T .

Discuss the case of elliptic T .

MATH 471

Problem Set 2

I. d_s denotes the **spherical metric** on $\mathbb{C} \cong \mathbb{R}^2$ (ie. the metric induced from the standard metric on S^2 via the stereographic projection).

1. Compute $d_s(z = 0, z = 1)$, $d_s(z = 1, z = i)$.
 - a) Compute the length l in the spherical metric of (i) the line segment connecting $z = 0$ to $z = 1$, (ii) the line segment connecting $z = 1$ to $z = i$, (iii) the circular arc in the first quadrant connecting $z = 1$ to $z = i$.
 - c) Explain why we have (i) $l = d_s(0, 1)$, (ii) $l \neq d_s(1, i)$, (iii) $l = d_s(1, i)$.
 - d) Explain (do not compute) which curve in \mathbb{R}^2 you will choose to compute $d_s(1, 2)$.
2. Show that the metric space (\mathbb{R}^2, d_s) is bounded but not compact.
3. Determine the group $\text{Mob}(\mathbb{C})$.

Show that $G = \{T \in \text{Mob}(\mathbb{C}) : T \text{ leaves } d_s \text{ invariant}\}$ is a group. Describe the elements of this group geometrically.

II. $l_{\mathbb{H}}$ is the hyperbolic length function.

1. Consider $\gamma_1(t) = te^{i\pi/4} + (1-t)e^{i3\pi/4}$ for $t \in [0, 1]$, $\gamma_2(t) = 2e^{i\theta}$, $\theta \in [\pi/4, 3\pi/4]$.
 - a) Compute $l_E(\gamma_i)$, $l_{\mathbb{H}}(\gamma_i)$.
 - b) Explain the results in terms of the Euclidean distance, Hyperbolic distance and the corresponding distance minimizing curves.
2. True or false ? Explain.
 - a) $\{d_{\mathbb{H}}(z_1, z_2) : z_1, z_2 \in \mathbb{H}\} = \{d_{\mathbb{H}}(i, z) : z \in \mathbb{H}\}$.
 - b) For all $z_1, z_2 \in \mathbb{H}$ and for $\gamma(t) = tz_1 + (1-t)z_2$, $t \in [0, 1]$ one has $l_E(\gamma) \leq l_{\mathbb{H}}(\gamma)$.
 - c) $\{(z_1, z_2) \in \mathbb{H} \times \mathbb{H} : z_1 \neq z_2, d_E(z_1, z_2) = d_{\mathbb{H}}(z_1, z_2)\} = \emptyset$.

MATH 471

Problem Set 3

d_E, d_S, d_H denote the **euclidean, the spherical and the hyperbolic metrics** on \mathbb{H} , respectively.

1. Show that for any $z \in \mathbb{H}$ and any $\epsilon > 0$, we have ϵ_1, ϵ_2 such that $D_H(z; \epsilon_2) \subset D_E(z; \epsilon_1) \subset D_H(z; \epsilon)$.
Conclude that $id : (\mathbb{H}, d_E) \rightarrow (\mathbb{H}, d_H)$ is a homeomorphism.
2. a) Show that for any curve γ in \mathbb{H} , we have $l_S(\gamma) \leq l_E(\gamma)$ and $l_S(\gamma) \leq l_H(\gamma)$.
b) Show that there is no such a relation between l_E and l_H .
3. Show that there exists no isometry $\phi : (\mathbb{H}, d_E) \rightarrow (\mathbb{H}, d_H)$.
4. Show that (\mathbb{H}, d_H) is a complete metric space.
5. a) Show that for a closed subset $X \subset \mathbb{H}$ and for $r \geq 0$, the set $X(r) = \{z \in \mathbb{H} : d(z, X) = r\}$ is closed.
b) For $X =$ the positive imaginary axis, determine $X(1)$.
(Hint : You may use Exercise 3.18).
c) Show that if $r > 0$ and if X is a hyperbolic line, then $X(r)$ does not contain any hyperbolic line.
6. a) Let $f : D \rightarrow Y$ be a continuous map into a topological space Y , which is invariant under $\text{Mob}(D)$. Determine $f(D)$.
b) Give examples of nonconstant functions $F : D \times D \rightarrow \mathbb{R}$, which are invariant under $\text{Mob}(D)$.
Conclude that $\text{Mob}(D)$ does not act doubly transitively on D .
7. a) Let $T : \mathbb{H} \rightarrow D$ be a linear fractional transformation. Sketch the images of the geodesic lines $L_1 = \{z \in \mathbb{H} : \text{Re}(z) = 0\}$, $L_3 = \{z \in \mathbb{H} : \text{Re}(z) = 1\}$ and the semicircle $L_2 : |z - 1/2| = 1/2, \text{Im}(z) > 0$ under T .
b) For $T' \in \text{Mob}(D)$ which maps $T(L_1)$ onto $T(L_3)$, sketch $T'(T(L_2)), T'(T(L_3))$.

MATH 471

Problem Set 4

For the definition of the **curvature of a metric** see p.108 in the textbook.

1. Compute the curvature of (\mathbb{H}, d_S) and of $(\mathbb{H}, d_{\mathbb{H}})$.
Deduce that (\mathbb{H}, d_S) and $(\mathbb{H}, d_{\mathbb{H}})$ are not isometric.
2. Problems 5.20 and 5.22 in the textbook.
3. Consider the hyperbolic triangle Δ in \mathbb{H} with vertices at $z_1 = -1 + i$, $z_2 = 1 + i$, $z_3 = iy$, $y > 1$.
 - a) Determine an upper bound for the area $A(\Delta)$.
 - b) Determine the minimum value of y (if exists) such that the area of the corresponding triangle satisfies $A(\Delta) \geq \pi/3$.
4. Consider $T \in Mob(\mathbb{H})$, $T(z) = \frac{i\bar{z} + i}{2i\bar{z} + i}$.
 - a) Determine the fixed points of T .
 - b) Find the hyperbolic line l such that $T(l) = l$.
5. a) Define rigorously the reflection $\rho_l : \mathbb{H} \rightarrow \mathbb{H}$ through a hyperbolic line l .
(Hint: Reduce to the case in which $l = \{z = iy : y > 0\}$).
 - b) Determine ρ_l for the hyperbolic line l of prob.4(b).
6. a) Define rigorously the translation $\tau_l : \mathbb{H} \rightarrow \mathbb{H}$ along a hyperbolic line l .
(Hint : First consider the case of $l = \{z = iy : y > 0\}$).
 - b) Determine τ_l for the hyperbolic line l of prob.4(b).
 - c) Show that the transformation T of prob.4 is of the form $T = \tau_l \circ \rho_l$.
7. Exercise 6.1 in the textbook.