MATH 252

EXERCISE SHEET 1

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1. Let ϕ be a scalar field and F be a vector field. Prove that $\nabla \circ (\phi F) = \phi \nabla \circ F + F \circ \nabla \phi$.

2. Let $F, G : \mathbb{R}^3 \to \mathbb{R}^3$ be two vector fields. Prove the following identities.

   $\nabla \circ (F \times G) = (\nabla \times F) \circ G - F \circ (\nabla \times G)$.

   $\nabla \times (\nabla \times F) = \nabla(\nabla \circ F) - \nabla^2 F$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Define $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Phi(p) = f(|p|)$. Show that there exists a scalar field λ such that $\nabla \Phi(p) = \lambda(p) p$ if $p \neq \mathbf{0}$.

4. Find an example of a scalar field ϕ and a vector field F neither of which is constant, for which $\nabla \circ (\phi F)$ is identically equal to $\phi \nabla \circ F$.

5. Let $F(x, y, z) = (x + xz^2)\vec{i} + xy\vec{j} + yz\vec{k}$. Find $\nabla \circ F$ and $\nabla \times F$.

6. Find a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\nabla \times F = y\vec{i} + x\vec{j}$.

7. Let

   $$x = ucos(v)$$
   $$y = usin(v)$$
   $$z = w$$

   where $u \geq 0$, $0 \leq v \leq 2\pi$ and $w \in \mathbb{R}$.

   (i) Find unit vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and show that they form an orthogonal curvilinear system.

   (ii) Find $\nabla^2 F$ if $F(x, y, z) = (x^2 + y^2)^{3/2} \arctan(\dfrac{y}{x}) + x^2 + y^2 + z^2$

8. Consider the family of curves $xy = c,\ c > 0$ in $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$.

   a) Find the orthogonal trajectories $v(x, y) = d$ of this family *.

   b) Write the operator ∇ in Ω , in the orthogonal coordinate system thus obtained *.

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EXERCISE SHEET 2

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1. Let $\mathbf{R}$ be the triangular region bounded bye the lines $y = 2x$, $x = 0$ and $y = 4$ and let $\mathbf{R}'$ be the rectangle whose sides are the lines $x = 0$, $x = 2$, $y = 0$ and $y = 4$. Suppose the partition $\mathbf{P}$ of $\mathbf{R}'$ consists of the squares whose sides are 1 unit long. If $f(x, y) = x + y$ for $(x, y) \in \mathbb{R}$, find

   (a) $\underline{S}(\mathbf{P}, f)$

   (b) $\overline{S}(\mathbf{P}, f)$

   (c) the Riemann sum $S(\mathbf{P}, f)$ of f that uses the midpoints of the rectangles of $\mathbf{P}$.

2. If f is a continuous on D, A is the area of D and $m \leq f(x, y) \leq M$ for all $(x, y) \in D$, show that

$$mA \leq \int\int_D f(x, y)dxdy \leq MA.$$

3. Compute $\displaystyle\int_1^4 \int_{\sqrt{y}}^2 \sin(\frac{x^3}{3} - x)dxdy$.

4. Find the area of the region bounded bye the curves $x = y^2$ and $x = 4y^2 - 3$.

5. Find the volume of the solid region bounded bye the paraboloid $z = 4 - x^2 - y^2$ and the planes $z = 3$ and $z = 0$.

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EXERCISE SHEET 3

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1. Find the area of the limacon $r = 2 - \sin\theta$. (Sketch the region)

2. Find the area of the region bounded by the graph of $r = 2\sin 3\theta$. (Sketch the region)

3. Change the order of the integration to write the following sum of iterated integrals

$$\int_0^{R/\sqrt{2}} \int_0^{\pi} f(x,y)dxdy + \int_{R/\sqrt{2}}^{R} \int_0^{\sqrt{R^2-x^2}} f(x,y)dydx$$

   as a single integral of the form

$$\int\int_D f(x,y)dxdy.$$

4. Compute the following integral:

$$\int_{-8}^{8} \int_{-\sqrt{64-x^2}}^{\sqrt{64-x^2}} (16 - 2\sqrt{x^2+y^2})dydx$$

5. Find the volume of the solid region above the $xy-$ plane bounded on the sides by the cylinder $x^2 + y^2 - 4x = 0$, above bye the cone $z^2 = x^2 + y^2$.

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EXERCISE SHEET 4

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1. The gamma function is defined as,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

   Show that $\Gamma(1/2) = \sqrt{\pi}$.

2. Evaluate the integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$$

3. Find $F'(x)$ if

$$F(x) = \int_1^\infty \frac{\cos(xu^2)}{u} du.$$

4. Evaluate

$$\int_0^a \int_0^b \exp(\max(b^2x^2, a^2y^2)) dy dx,$$

   where a and b are positive real numbers.

5. What relation must hold between the constants a, b and c to make

$$\int_{-\infty}^\infty \int_{-\infty}^\infty e^{(ax^2+2bxy+cy^2)} dx dy = 1?$$

   (Hint: Make a substitution s and t so that $ax^2 + 2bxy + cy^2 = s^2 + t^2$.)

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EXERCISE SHEET 5

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6. Determine the arc length of the curve $x = e^t \cos t$, $y = e^t \sin t$, $z = 0$ between $t = 0$ and $t = 1$.

7. Show that the curve $x = t$, $y = 2t^2$, $z = t^3$ intersects the plane $x + 8y + 12z = 162$ at right angles.

8. Find the value of $\oint [(3x + 4y)dx + (2x + 3y^2)dy]$ around the circle $x^2 + y^2 = 4$.

9. Find $\int F dR$ from $(1, 0, 0)$ to $(1, 0, 4)$, if $F = (x, -y, z)$

   (a) along the line segment joining these two points,

   (b) along the helix $x = \cos(2\pi t)$, $y = \sin(2\pi t)$, $z = 4t$.

10. Evaluate

$$\oint [(y + yz \cos xyz)dx + (x^2 + xz \cos xyz)dy + (z + xy \cos xyz)dz]$$

along the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, $z = 1$.

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EXERCISE SHEET 6

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11. Let $F = (2xyz + z^2 - 2y^2 + 1)\vec{i} + (x^2z - 4xy)\vec{j} + (x^2y + 2xz - 2)\vec{k}$ and let C be the directed curve which is the broken line obtained as the line segment from the origin to $(1,2,3)$ followed by the line segment from $(1,2,3)$ to $(3,4,5)$. Find $\int_C FdR$.

12. Evaluate $\int_{(0,0)}^{(1,2)}[(15x^4 - 3x^2y^2)dx - 2x^3ydy]$ along the path $2x^4 - 6xy^3 + 23y = 0$.

13. Find the area inside the loop of Descartes'folium $x = \frac{t}{1+t^3},\ y = \frac{t^2}{1+t^3},\ 0 \leq t < \infty$.

14. Evaluate the line integral $\int_C xyzds$, where C is the part of the helix $x = a\cos t,\ y = a\sin t,\ z = bt$ from $(0,0,0)$ to $(-a, 03b\pi)$.

15. Let C be the boundary of the region bounded by $x = y^2,\ x = 1 + y^2,\ xy = 2,\ xy = 3$ which is oriented counterclockwise. Evaluate $\oint_C[(e^{x^3} - \frac{2}{3}y^3)dx + (\sin y^3 + \frac{1}{2}x^2)dy]$.

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EXERCISE SHEET 7

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16. Let $f(x,y)=\ln(x^2+y^2)$ and let C be the circle $x^2+y^2=a^2$. Evaluate the flux integral

$$\oint_C \nabla f \cdot n ds,$$

where n is the unit normal vector to C .

17. (a) Find the area enclosed by the curve

$$r=(\frac{2\cos t-\sin t}{2})\vec{i}+(\sin t)\vec{j},\quad 0\le t\le 2\pi.$$

(b) Find an equation for the curve in Cartesian coordinates and use the equation to identify the curve.

18. Let P, Q have continuous first partial derivatives with $P_y = Q_x$ except at (4.0), $(0,0)$, $(-4,0)$. Let C_1 denote $(x-2)^2+y^2=9$, C_2 denote $(x+2)^2+y^2=9$, C_3 denote $x^2+y^2=25$ and C_4 denote $x^2+y^2=1$. Suppose that

$$\oint_{C_1} Pdx+Qdy=11,\ \oint_{C_2} Pdx+Qdy=9,\ \oint_{C_3} Pdx+Qdy=13.$$

Find $\oint_{C_4} Pdx+Qdy$.

19. Find the area of the portion of the surface $x=9-y^2-z^2$ that lies above the ring $1\le y^2+z^2\le 9$ in the $yz-$ plane.

20. Integrate $g(x,y,z)=xyz$ over the surface of the rectangular solid bounded by the planes $x=\pm a$, $y=\pm b$ and $z=\pm c$.

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EXERCISE SHEET 8

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21. Compute the surface integral of $F = (x, 0, 0)$ over the triangle with vertices $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$ taking the normal on the side away from the origin.

22. Given $F = (x, -y, 0)$. Find the value of $\int\int F \cdot ndS$ over the closed surface bounded by the planes $z = 0$, $z = 1$ and the cylinder $x^2 + y^2 = a^2$, where n is the unit outward normal.

23. Let $E = -grad(|R|^{-1})$ where $R = (x, y, z)$.

    (a) Show that $E = R/|R|^3$.

    (b) Find $\int_C EdR$, when C is the line segment joining the points $(0, 1, 0)$ and $(0, 0, 1)$.

    (c) Compute $\int\int_{S_1} E \cdot ndS$, when S_1 is the sphere $x^2 + y^2 + z^2 = 9$.

    (d) Evaluate $\int\int_{S_2} E \cdot ndS$, when S_2 is a cube with edges one unit long, centered at the origin.

    (e) Give, if possible, an example of a sphere S with positive redius such that $\int\int_S E \cdot ndS = 0$.

24. Find the area of the part of the graph of $z = x^2 - y^2$ whose projection onto $xy-$ plane is $x^2 + y^2 \leq 1$.

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EXERCISE SHEET 9

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25. Evaluate $\int\int_C ydA$, where C is the upper half of the cardioid disk $r \leq 1+\cos\theta$.

26. Find the volume lying between the paraboloids $z = x^2+y^2$ and $3z = 4-x^2-y^2$.

27. Find the volume lying inside both the sphere $x^2+y^2+z^2 = 2a^2$ and the cylinder $x^2+y^2 = a^2$.

28. Find the area of the region in the first quadrant bounded by the curves $xy = 1, xy = 4, y = x$ and $y = 2x$.

29. $\int\int\int_R (1+2x-3y)dV$ over the box $-a \leq x \leq a,\ -b \leq y \leq b,\ -c \leq z \leq c$.

30. $\int\int\int_D (3+2xy)dV$ over the solid hemispherical dome D given by $x^2+y^2+z^2 \leq 4$ and $z \geq 0$.

31. $\int\int\int_R (xy+z^2)dV$ over the set $0 \leq z \leq 1-|x|-|y|$.

32. $\int\int\int_R \sin(\pi y^3)dV$ over the pyramid with vertices $(0,0,0), (0,1,0), (1,1,0)$, $(1,1,1)$ and $(0,1,1)$.

33. $\int\int\int_R \frac{1}{(x+y+z)^3}dV$ over the region bounded by the six planes $z = 1, z = 2, y = 0, y = z, x = 0$ and $x = y+z$.

34. Find the volume of the region lying inside the cylinder $x^2+4y^2 = 4$, above the $xy-$ plane and below the plane $z = 2+x$.

    In exercises 11 and 12 below express the given integral and sketch the region over which integral is taken. Write the integrals so that the outermost integral is with respect to x and the innermost with respect to z .

35. $\int_0^1 dz \int_0^{1-z} dy \int_0^1 f(x,y,z)dx$

36. $\int_0^1 dz \int_0^1 dx \int_0^{x-z} f(x,y,z)dy$

37. Describe the set of point in $3-$ space that satisfy the following equations in spherical coordinates

    $(a)\theta = \pi/2 \qquad (b)\phi = \pi/2 \qquad (c)\rho = 4 \qquad (d)\rho = 2\cos\phi$.

    In exercise $14-16$ find the volumes of the indicated regions

38. Inside the cone $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 = a^2$.

39. Between the paraboloids $z = 10-x^2-y^2$ and $z = 2(x^2+y^2-1)$.

40. Above the $xy-$ plane, inside the cone $z = 2a-\sqrt{x^2+y^2}$ and inside the cylinder $x^2+y^2 = 2ay$.

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EXERCISE SHEET 10

I. Change the order of the integration

41. $\int_0^1 \int_{2x}^{x} f(x,y)dydx$

42. $\int_0^a \int_{\frac{a^2-x^2}{2a}}^{\sqrt{a^2-x^2}} f(x,y)dydx$

43. $\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{4ax} f(x,y)dydx$

44. $\int_0^2 \int_{x^2}^{x^3} f(x,y)dydx$

45. $\int_0^{R/\sqrt{2}} \int_0^x f(x,y)dydx + \int_{R/\sqrt{2}}^R \int_0^{\sqrt{R^2-x^2}} f(x,y)dydx$

II. Sketch the region D and then evaluate

46. $\int_0^\pi \int_0^{1+\cos x} y^2 \sin x dydx$

47. $\int_{-\pi/2}^{\pi/2} \int_0^{3\cos y} x^2 \sin^2 y dxdy$

III. Change the variables as indicated and evaluate

48. $\int_0^2 \int_0^x f(\sqrt{x^2+y^2})dydx$ (use polars)

49. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}dydx$ (use polars)

50. $\int\int_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}dxdy$, D is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ - use $x = ar\cos\theta$, $y = br\sin\theta$.

51. $\int_0^c \int_{ax}^{bx} f(x,y)dydx$, $(0 < a < b,\ c > 0)$ - use $u = x+y,\ uv = y$.