Graduate Preliminary Exam

Ordinary Differential Equations - Fall 2008

1. Consider the system

$$x' = -3x - 2y + \sin(t),$$
$$y' = 2x - 3y + \cos(t). \quad (1)$$

   (a) Evaluate the transition matrix $X(t,s)$ of the associated homogeneous system. Show that $\limsup_{t\to\infty} \|X(t,s)\| = 0$;

   (b) Find the general solution $x(t, t_0, x_0)$ of the system;

   (c) Show that all solutions are bounded on $[0,\infty)$ functions;

   (d) Show that there exists a unique solution bounded on R;

   (e) Prove that the bounded solution is $2\pi-$periodic function.

   (f) Prove that each solution of the system is uniformly asymptotically stable.

2. Estimate $|x(t,0,x_0,x_0^1)|, |x'(t,0,x_0,x_0^1)|$ for $t \in [0,T],\ T < \infty$ if

$$x(t) = x(t,0,x_0,x_0^1),\ x(0) = x_0,\ x'(0) = x_0^1$$

   is a solution of equation $x'' + \sin x = 0$. (Take $x_0 = 0.01, x_0^1 = -0.02, T = 10$).

   Hint: Use differentiability of solutions in initial value.

3. Assume that $u(t) \geq 0, v(t) > 0$, are continuous functions on $[t_0 - T, t_0], t_0 \in R, T > 0$. Prove that the inequality

$$u(t) \leq c + \int_t^{t_0} u(s)v(s)ds, t \leq t_0$$

implies

$$u(t) \leq c e^{\int_t^{t_0} v(s) ds}$$

where $c \geq 0$ is constant.

4. Consider the following Abel's equation

$$y' = \sin(t) - y^3 \qquad (1)$$

where $t, y \in R$. Prove that as t increases, each solution of (1) is attracted into the strip $|y| < 1 + \epsilon$, where ϵ is a fixed positive number, in a uniformly bounded time interval.