September, ?? 2008

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. Let $P(x)$ be the polynomial of lowest degree that interpolates the function $f(x)$ the given data $f(0) = -1,\ f(1) = 0,\ f'(0) = -2,\ f'(1) = 5,\ f''(1) = 10$.

   (a) What is the degree of $P(x)$? Find the explicit form of the polynomial $P(x)$ using Newton's divided differences.

   (b) Estimate the error of the interpolation $f(x) - P(x)$ for the $f(x) = \sin x$ at the point $\bar{x} = 0.5$.

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2. Consider the integral $\int_a^b f(x)dx$.

(a) Divide the interval $[a, b]$ into N uniform intervals and write down the composite trapezoidal rule $I(N)$.

(b) Suppose we have to approximate solutions $I(N)$ and $I(N/3)$ which are obtained by the composite trapezoidal rule. What is the order of accuracy of $I(N)$ and $I(N/3)$?

(c) Construct an approximation of higher order than $I(N)$ and $I(N/3)$ using only the values of $I(N)$ and $I(N/3)$.

(d) What is the order of accuracy of this higher order approximation?

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3. Given a matrix $A \in \mathbb{R}^{n,n}$ and an eigenvector z of A associated with the eigenvalue λ. Thus, zeros of the function

$$F\begin{pmatrix} z \\ \lambda \end{pmatrix} = \begin{pmatrix} Az - \lambda z \\ 1 - \frac{1}{2}||z||_2 \end{pmatrix}$$

provide eigenvectors and eigenvalues of A.

(a) Formulate the Newton iteration for $F\begin{pmatrix} z \\ \lambda \end{pmatrix} = 0$.

(b) Give the conditions for the current iterate $z^{(k)}$ and $\lambda^{(k)}$ that guarantee that the next Newton iteration can computed.

(c) What is the computational complexity of each Newton iteration?

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4. Show that in the three-term recurrence relation for orthogonal polynomials q_k in

$$q_k(x) = (x - a_k)q_{k-1}(x) - b_k q_{k-2}(x), \quad k \geq 2$$

is given by $b_k =< q_{k-1}, q_{k-1} > / < q_{k-2}, q_{k-2} >$.

Hint: Show by $p(x) = \sum_{j=0}^{n} < p, p_j > p_j(x)$ that

$$x q_{k-2}(x) = \sum_{i=0}^{k-1} a_i q_i(x).$$
