September, ?? 2008

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Let $D = diag(d_1^2, d_2^2)$ be a regular diagonal 2×2 matrix with $d_1, d_2 > 0$.

   (a) Show that it induces a vector norm $||\cdot||_D$ on $\mathbb{R}^2$ through

   $$||x||_D^2 = x^T Dx.$$

   (b) For any $A \in \mathbb{R}^{2\times 2}$ give a formula for a matrix $B \in \mathbb{R}^{2\times 2}$ such that

   $$||A||_D = ||B||_2.$$

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2. Let A be a symmetric matrix and let λ and x be an eigenvalue-eigenvector pair for$A$ with $||x||_2 = 1$. Let P be an orthogonal matrix for which $Px = e_1 = [1, 0, \cdots, 0]^T$ and consider the similar matrix $B = PAP^T$.

   (a) Show that the first row and column of B are zero except for the diagonal element which equals λ.

   (b) For the matrix

$$A = \begin{bmatrix} 2 & 10 & 2 \\ 10 & 5 & -8 \\ 2 & -8 & 11 \end{bmatrix}$$

   with $\lambda = 9$ as an eigenvalue with associated eigenvector $x = [\frac{2}{3}, \frac{1}{3}, \frac{2}{3}]^T$. Produce a Householder matrix P for which $Px = e_1$ and then produce $B = PAP^T$.

   (c) By using the matrix B in part (b) find two other eigenvalues of A.

3. Suppose the matrix $A \in \mathbb{C}^{m \times n}, m \geq n$ has full rank.

(a) Use the normal equations to derive the compact formula for the optimal solution x of the least squares problem

$$\min_{x \in \mathbb{C}^n} ||Ax - b||_2$$

in terms of the singular value decomposition of A.

(b) When the matrix A is rank deficient (or nearly so, i.e. $\sigma_n << \sigma_1$), it is often appealing to replace the standard least squares problem by the regularized problem

$$\min_{x \in \mathbb{C}^n} ||Ax - b||_2^2 + \lambda ||x||_2^2$$

for some real constant λ. Here σ_i denotes the singular values of A.

Show that x solves the regularized problem

$$(A^* A + \lambda I)x = A^* b.$$

(c) Use the equation (b) to derive an expression for the solution x to the regularized problem in terms of singular value decomposition.

4. Consider the sparse linear system of equations

$$(I + A)x = b$$

with $b \in \mathbb{R}^n$, $A := \frac{1}{4}(P_1 + P_2 + P_3)$, where $P_i \in \mathbb{R}^{n\times n}, i = 1, 2, 3$ are arbitrary permutation matrices. A permutation matrix is a square matrix whose entries are all 0's and 1's, with exactly one 1 in each row and exactly one 1 in each column and $P^{-1} = P^T = P$.

(a) Show that $||Px||_2 = ||x||_2$ for any permutation matrix $P \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$.

(b) Show that $I + A$ is regular and give upper bounds for $||A||_2$, $||I + A||_2$, $||(I + A)^{-1}||$, and $\mathrm{cond}_2(\mathrm{I} + \mathrm{A})$.

Hint: You may use the perturbation lemma

$$X \in \mathbb{R}^{n\times n}, \quad ||X||_2 < 1, \quad ||(I + X)^{-1}|| \leq \frac{1}{1 - ||X||}.$$

(c) Show that the fixed point iteration

$$x^{(k+1)} = b - Ax^{(k)}, \quad k = 0, 1, 2, \ldots$$

is consistent with $(I + A)x = b$.

(d) Show that the iteration in (c) converges linearly w.r.t Euclidean norm and give an estimate for the converge rate.