METU - Department of Mathematics

Graduate Preliminary Exam

Geometry

September 18, 2008

1. a) Show that the 1-form

$$w = \frac{ydx - xdy}{x^2 + y^2}$$

is a closed 1-form on $\mathbb{R}^2 \setminus \{(0,0)\}$.

b) Evaluate $\int_{S^1} w$ where S^1 is the unit circle in $\mathbb{R}^2$.

c) Is the form w exact? Prove your answer.

d) Let

$C_1 = S^1 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ and $C_2 = \{(x,y) \in \mathbb{R}^2 | (x-2)^2 + y^2 = 1\}$.

Determine if there is a diffeomorphism $\varphi : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$ such that $\varphi(C_1) = C_2$.

What happens if we let φ be a diffeomorphism of $\mathbb{R}^2$?

2. Consider

$$\mathbb{RP}^2 = \frac{S^2}{(x,y,z) \sim (-x,-y,-z)}$$

as the quotient space of S^2 by the equivalence relation σ, where $(x,y,z) \sim (x',y',z')$ if and only if $(x',y',z') = (x,y,z)$ or $(-x,-y,-z)$, define the map $F : \mathbb{RP}^2 \to \mathbb{R}^5$ by setting $[x:y:z] \longrightarrow F([x:y:z]) = (x^2, y^2, xy, xz, yz)$ where $[x:y:z]$ denotes the equivalence class of the point $(x,y,z) \in S^2$ (S^2 is the unit sphere in $\mathbb{R}^3$).

a) Prove that F is $1-1$.

b) Prove that F is a homeomorphism onto its image.

c) Prove that F is an immersion.

d) Conclude that F is an embedding (smooth).

3. Show that there is no smooth map $f : \sum \to S^1$ such that $f|_{\varphi\Sigma = C} : C \to S^1$ is a diffeomorphism, where S^1 is the unit circle in $\mathbb{R}^2$ and Σ is the surface with boundary below:

Figure.

4. a) Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be a smooth function such that $|f(x,y)| \in \frac{1}{\sqrt{r}}$, where $r = x^2 + y^2$. If S^1_R denotes the circle of radius R and with center at the origin show that

$$|\int_{S^1_R} f(x,y)dx| \leq 2\pi\sqrt{R} \quad \text{and}$$

$$\text{similarly} \quad |\int_{S^1_R} f(x,y)dy| \leq 2\pi\sqrt{R}.$$

b) Let $w = \rho(x,y)dx + q(x,y)dy$, where

$$|\rho(x,y)| \leq \frac{1}{\sqrt{r}} \quad \text{and} \quad |q(x,y)| \leq \frac{1}{\sqrt{r}}$$

for any $(x,y) \neq (0,0)$ ($p(x,y), q(x,y)$ smooth functions or $\mathbb{R}^2 \setminus \{(0,0)\}$).

Show that $|\int_{S^1_R} w| \leq 4\pi\sqrt{R}$, $R > 0$.

c) Assume now that w is also closed. Show that $|\int_{S^1} w| \leq 4\pi\sqrt{R}$ for any R (Use Stokes' Thm).

d) Conclude that $\int_{S^1} w = 0$.