**METU - Mathematics Department**
**Graduate Preliminary Exam**

**Complex Analysis**

**Duration : 3 hours** **September 2006**

1. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{at } (0,0). \end{cases}$$

   Show that

   a) $f(x,y)$ is a continuous function of the variables $x$ and $y$.

   b) The functions $u(x,y) = Re(f(x,y))$ and $v(x,y) = Im(f(x,y))$ satisfy the Cauchy-Riemann equations at $z = 0$.

   c) $f'(z)$ does not exist at $z = 0$.

   Why does the conclusion in (c) not contradict part (b) ?

2. Let $f(z)$ be a continuous function on the unit circle $S^1 = \{z : |z| = 1\}$. Show that the function

$$F(z) = \int_{S^1} \frac{f(\xi)}{(\xi - z)} d\xi$$

   is analytic on $D = \{z : |z| < 1\}$ and that

$$F'(z) = \int_{S^1} \frac{f(\xi)}{(\xi - z)^2} d\xi.$$

3. Explain how to choose a branch of $f(z) = \sqrt{z^2 - 1}$ which is analytic in $\mathbb{C} - [-1, 1]$.

   a) Using this branch and the residue at infinity, compute $\displaystyle\int_\Gamma f(z)dz$ where $\Gamma$ is the circle $|z| = \rho > 1$.

   b) Compute the improper integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^2-1}}$ using the complex integral $\displaystyle\int_\Gamma \frac{dz}{f(z)}$ where $\Gamma$ is given as

   (Hints :
   1) Residue at infinity is defined by res$(g(z), \infty)$ = -res$(g(1/t)/t^2, 0)$.
   2) The binomial series is given by $(1+z)^\alpha = \sum c_n z^n$ where $c_n = \dfrac{\alpha.(\alpha-1)...(\alpha-n+1)}{n!}$.)

4. Let $f(z)$ be an analytic function in the unit disk $D = \{z : |z| < 1\}$ and suppose that $|f(z)| \leq 1$ in $D$. Prove that if $f(z)$ has at least two fixed points, then $f(z) = z$ for all $z \in D$.

(Hint : Using a suitable automorphism of the disk reduce to the case where one of the fixed points is $0 \in D$ so that $g(z) = f(z)/z$ defines an analytic function on $D$.)