PRELIMINARY EXAM PROBLEMS
Differential Equations (PDE), 2005/2, 3 hours

1. Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad x^2 + y^2 < 1,$$

$$u = y^4, \quad x^2 + y^2 = 1.$$

2. (a). Find, for all positive and negative values of a constant λ, the real solutions of the equation

$$\frac{\partial^2 z}{\partial x^2} = c^2 \frac{\partial z}{\partial t}$$

that are of the form $z = \mathrm{e}^{\lambda t}\phi(x)$.

(b). If c is not an integer multiple of π, show that there exists a solution of this equation which remains finite as $t \to \infty$, which is zero when $x = 0$, and which assumes the value $\mathrm{e}^{-\mathrm{t}}$ when $x = 1$. Find this solution.

3. (a). Let $u(x,t)$ be a solution of the equation

$$u_t - ku_{xx} = F(x,t),\ k > 0, \tag{1}$$

for $\{(x,t) \mid 0 < x < L,\ t > 0\}$, where L is a fixed positive number, and $u(x,t)$ is continuous in $\{(x,t) \mid 0 \leq x \leq L,\ t \geq 0\}$. Prove that the maximum of $u(x,t)$ is attained at $t = 0$, or $x = 0$, or $x = L$, if $F(x,t)$ is negative valued in $\{(x,t) \mid 0 < x < L, t > 0\}$.

(b). Construct a counter example if $F(x,t)$ in (1) is positive in the region.

4. The function $\frac{-1}{2\pi}K_0(\alpha r)$ is a fundamental solution for the equation

$$\nabla^2 u - \alpha^2 u = 0 \quad in \quad \Omega,$$

where α is a constant, $\Omega \subset R^2$ and $K_0(\alpha r)$ is the zero order modified Bessel function of the second kind, r is the distance from a fixed point (ξ, η) to any point (x, y) in Ω.

Prove that the Green's function for the equation above defined by

$$\nabla^2 G - \alpha^2 G = \delta(x - \xi)\delta(y - \eta) \quad in \quad \Omega,$$

$$G = 0 \quad on \quad \partial\Omega,$$

is unique, where δ and ∇^2 denote the Dirac Delta and Laplace operators respectively.