

**PRELIMINARY EXAM PROBLEMS**  
**Differential Equations (ODE), 3 hours, 2005/2**

1. Consider the IVP

$$x_1' = -\frac{1}{t}x_2 + \sin t,$$

$$x_2' = \frac{1}{t}x_1 + \cos t,$$

$$x_1(1) = x_2(1) = 0.$$

Show that the IVP has a unique solution defined on  $(0, \infty)$ .

2. Let  $p(t)$  be continuous function defined on  $[1, \infty)$  such that

$$\int_1^\infty |p(t) - c| dt < \infty, \quad c > 0.$$

(a). Show that all solutions of

$$y'' + p(t)y = 0 \tag{1}$$

are bounded on  $[1, \infty)$ . (Hint: rewrite the equation in the form  $y'' + cy = (c - p(t))y$ ).

(b). What can you say about the stability of the zero solution?

(c). Show that all solutions of (1) need not to be bounded if  $c = 0$ . (Hint:  $p(t) = \frac{a}{t^2}$ ).

3. Let  $A(t)$  be a continuous matrix for all  $t \in \mathbb{R}$ . Let  $P(t)$  be the matrix solution of

$$X' = A(t)X.$$

Show that  $P(t)P^{-1}(s) = P(t-s)$  for all  $t, s \in \mathbb{R}$ , if and only if  $A(t)$  is a constant matrix.

4. Consider the following scalar equation

$$x' = c(t)x, \tag{2}$$

where function  $c(t) : \mathbb{R} \rightarrow \mathbb{R}$  is defined in the following way:

$$c(t) = \begin{cases} t, & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1-t, & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

and  $c(t)$  is 1-periodic.

(a). Prove that (2) does not have a nontrivial 1-periodic solution.

(b). Does the equation have a nontrivial solution with another period?