METU - Mathematics Department
Graduate Preliminary Exam

Geometry

Duration : 3 hours **Fall 2005**

1. **a)** Show that a one-to-one immersion of a **compact** manifold is an imbedding.

   b) Explain, in full details, why the map $\phi : (-\pi, \pi) \to \mathbb{R}^2$, $\phi(s) = (\sin(2s), \sin(s))$ shows that the conclusion in part (a) is false if X is not compact.

2. Let $SL_n(\mathbb{R})$ denote the $n \times n$ real matrices with determinant 1.

   a) Show that $SL_n(\mathbb{R})$ is a submanifold of the $n \times n$ matrices $M_n(\mathbb{R})$.

   b) Show that the tangent space to $SL_n(\mathbb{R})$ at the identity matrix I is $T_I SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : trace(A) = 0\}$.

3. **a**) What is meant by an orientation on a manifold ?

   b) Show that $S^n = \{\overline{x} \in \mathbb{R}^{n+1} : |\overline{x}| = 1\}$ is an oriented manifold, by defining an orientation on it.

   c) Show that the antipodal map $S^n \to S^n$, $\overline{x} \mapsto -\overline{x}$ is orientation preserving if and only if n is odd.

   d) Using (c), or otherwise show that $\mathbb{R}P^n$ is orientable if and only if n is odd.

4. **a)** Show that $X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ is a closed submanifold of $\mathbb{R}^3$.

   b) Verify that the restriction $\omega|_X$ of $\omega = \dfrac{xdy - ydx}{x^2 + y^2}$ is a closed 1-form on X.

   c) Calculate $\displaystyle\int_S \omega|_X$, where S is the circle $\{(x, y, 3) : x^2 + y^2 = 1\} \subset X$.
   Is $\omega|_X$ an exact form ? Why ?

   d) Consider the mapping $\Psi : \mathbb{R}^2 \to X$, $\Psi((s, t)) = (\cos(s), \sin(s), t)$. Show that Ψ is a differentiable map and that the form $\Psi^*(\omega|_X)$ is exact.