

**PRELIMINARY EXAMINATION**  
**ALGEBRA II**  
**Fall 2005**  
**September 16<sup>th</sup>, 2005**

*Duration: 3 hours*

1. Let  $f(x) = x^3 - 2x - 2 \in \mathbb{Q}[x]$ . Let  $K = \mathbb{Q}(\alpha)$  where  $\alpha$  is a real root of  $f$ , and let  $F$  be the Galois closure of the extension  $K/\mathbb{Q}$ .
  - a) Determine the group of  $\mathbb{Q}$ -automorphisms of  $K$ .
  - b) Determine the Galois group  $G(F/\mathbb{Q})$ .
  - c) Determine the Galois group  $G(F/K)$ .
  
2. Let  $K$  be a field of characteristic  $p$  (where  $p$  is a prime number). Let  $K^p = \{b^p | b \in K\}$ .
  - a) Show that  $K^p$  is a subfield of  $K$  and  $K/K^p$  is an algebraic extension.
  - b) Let  $a \in K, a \notin K^p$ . Prove that  $[K^p(a) : K^p] = p$ .
  
3. Let  $R$  be a principal ideal domain,  $M$  a free  $R$ -module, and  $S$  a submodule of  $M$ .  $S$  is called a pure submodule if
$$\text{whenever } ay \in S \text{ (with } a \in R \setminus \{0\}, y \in M), \text{ then } y \in S.$$
  - a) Show that  $\{0\}$  and  $R$  are the only pure submodules of  $R$ , considered as an  $R$ -module)
  - b) Find a proper, nontrivial pure submodule of  $R \oplus R$  (considered as an  $R$ -module).
  - c) Let  $N$  be a torsion-free  $R$ -module and  $\varphi : M \rightarrow N$  be an  $R$ -module homomorphism. Prove that  $\text{Ker}\varphi$  is a pure submodule of  $M$ .
  
4. Let  $R$  be a commutative ring with identity. Prove that every submodule of  $R$  is free iff  $R = \{0\}$  or  $R$  is a principal ideal domain. (**Warning:** To prove that  $R$  is a PID, you have to show  $R$  is an integral domain first.)