

PRELIMINARY EXAM PROBLEMS

Differential Equations (ODE), 2004/2

1. Consider the following IVP

$$y' = y \cos(x^2 + y^2), \quad y(0) = 1.$$

where $y, x \in \mathbb{R}^1$.

- Applying the "existence-uniqueness" theorem, determine a *specific* interval on which a unique solution is sure to exist.
 - Determine the *largest* possible interval (α, β) on which the solution is defined.
 - Explain why the solution of the IVP is always positive.
 - Is the solution strictly increasing over its interval of definition? Why? Why not?
2. a) Let $y(t)$ be a solution of $y'' - e^{-t}y = 0$.
Show that $y(t)$ can not vanish twice.

b) Prove that every solution of $y'' + (1 + a(t))y = 0$ has infinitely many zeros, if

$$\lim_{t \rightarrow \infty} a(t) = 0.$$

3. Suppose that all solutions of $y'' + a(t)y = 0$ are bounded. Show that if $\int_0^\infty |b(t)|dt < \infty$, then all solutions of $y'' + (a(t) + b(t))y = 0$ are also bounded.
4. Using Lyapunov function show that the zero solution of the system

$$\begin{aligned} x_1' &= -2x_1x_2^2 - x_1^3, \\ x_2' &= -x_2 + x_1^2x_2 \end{aligned} \tag{1}$$

is uniform asymptotically stable.