Graduate Preliminary Examination

Algebra II

22.9.2004: 3 hours

Problem 1. Let $K_0 = \mathbb{F}_{11}[X]/(X^2+1)$ and $K_1 = \mathbb{F}_{11}[Y]/(Y^2+2Y+2)$.

(a) Show that the K_i are fields for $i = 0, 1$.

(b) Find the orders of the K_i for $i = 0, 1$.

(c) Either exhibit an isomorphism of K_0 and K_1, or show that they are not isomorphic.

Problem 2. Show that the sum of all elements of a finite field is zero, except for $\mathbb{F}_2$.

Problem 3. Let L/K be a field-extension, and let α be algebraic over K with minimal polynomial f. Let $M = K(\alpha) \otimes_K L$. We know that M is a vector-space over L.

(a) Exhibit an embedding of L in M (as vector-spaces over L).

(b) Exhibit an embedding ι of L in M and a multiplication $\cdot$ on M such that the following conditions hold:

- M is a commutative ring with identity;
- ι is a ring-homomorphism;
- if $m \in M$ and $\ell \in L$, then $\iota(\ell) \cdot m$ is the product ℓm given by the vector-space structure.

(c) Show that $L[X]/(f)$ and $K(\alpha) \otimes_K L$ are isomorphic as rings.

Problem 4. Let R be a ring with 1. If M is an R-module, the **uniform dimension** of M ($\operatorname{ud} M$) is the largest integer n such that there is a direct sum $M_1 \oplus \ldots \oplus M_n \subseteq M$ with all the M_i non-zero. If no such integer exists then we say that $\operatorname{ud} M = \infty$. If $M \subseteq N$ are R-modules, M is said to be **essential** in N if every non-zero submodule of N has non-zero intersection with M. Suppose the $\operatorname{ud} M < \infty$ and $M \subseteq N$. Prove that M is essential in N if and only if $\operatorname{ud} M = \operatorname{ud} N$