Graduate Preliminary Examination
Differentiable Manifolds
Duration: 3 hours

September 26, 2003

1. We identify $\mathbb{R}^4$ with the set of 2×2 real matrices.

(5 pts.) (a) Show that the set $SL(2, \mathbb{R})$ of 2×2 real matrices whose determinant is equal to 1 is a submanifold of $\mathbb{R}^4$. What is its dimension?

(5 pts.) (b) Prove that the tangent space to $SL(2, \mathbb{R})$ at the identity matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, may be identified with the set of matrices of zero trace.

2. (3 pts.) (a) Show that the 1-form $\omega = \dfrac{xdy - ydx}{x^2 + y^2}$ defined on $\mathbb{R}^2 - \{(0,0)\}$ is closed.

(3 pts.) (b) Calculate the integral $\displaystyle\int_{S^1} \omega$, where S^1 is the unit circle in $\mathbb{R}^2$.

(4 pts.) (c) Let Σ be the smooth surface shown below with boundary C. Prove that there is no smooth map $\phi : \Sigma \to S^1$ such that $\phi_{|C} : C \to S^1$, the restriction of ϕ to the boundary C, is a diffeomorphism.

3. Let $f : X \to Y$ is a smooth map between manifolds, f^* is the induced map between the algebras of differential forms of X and Y and d is the exterior derivative.

(5 pts.) (a) Prove that $d \circ f^* = f^* \circ d$.

(5 pts.) (b) If $X = \partial W$ for some compact smooth manifold W, and ω is a closed n-form on Y with $n = \dim X$, then show that

$$\int_X f^*(\omega) = 0.$$

4. (10 pts.) A curve in a manifold X is a smooth map $t \mapsto c(t)$ of an interval of $\mathbb{R}^1$ into X. The velocity vector of the curve c at time t_0 - denoted simply by $\frac{dc}{dt}(t_0)$ is defined to be the vector $dc_{t_0}(1) \in T_{x_0}X$, where $x_0 = c(t_0)$ and $dc_{t_0} : \mathbb{R}^1 \to T_{x_0}X$ is the differential of c at t_0. In case $X = \mathbb{R}^k$ and $c(t) = (c_1(t), \cdots, c_k(t))$ in coordinates, check that

$$\frac{dc}{dt}(t_0) = (c_1'(t), \cdots, c_k'(t)).$$

Prove that any vector in T_xX is the velocity vector of some curve in X, and conversely.