

METU - Mathematics Department
Graduate Preliminary Exam-Spring 2008

Topology

1. Consider the real line \mathbb{R} with the usual topology. Let \sim be the equivalence relation defined by

$$x \sim y \text{ if and only if } x - y \in \mathbb{Q}.$$

Show that the quotient space \mathbb{R}/\sim has uncountable number of elements and that its topology is trivial.

2. A **discrete valued map** on a topological space X is a continuous map $X \rightarrow D$ into a discrete topological space D . **Show that**
- a) X is connected if and only if every discrete valued map on X is constant.
 - b) the statement " $d(p) = d(q)$ for every discrete valued map d on X " defines an equivalence relation on X and that the corresponding equivalence classes are closed subsets of X .

3. Let X and Y be two topological spaces and let $X \times Y$ be given the product topology.

a) Suppose K is a compact subset of X and $A \subset X \times Y$ is an open set such that for some $y \in Y$, $K \times \{y\} \subset A$. Show that y has a neighborhood $U \subset Y$ such that $K \times U \subset A$.

b) (i) Suppose X is compact. Prove that the projection $\pi : X \times Y \rightarrow Y$ is a closed map.

(ii) Give an example to show that in (i) the *compactness* assumption is essential.

4. Let X be a Hausdorff topological space and let $X^* = X \cup \{\infty\}$, where ∞ is an ideal point not in X . Consider the following collection Ω^* of subsets of X^*

(i) open sets in X

(ii) sets of the form $X^* - S$ where S is a compact subset of X .

Prove the following statements for the topological space (X^*, Ω^*) (**do not prove** that Ω^* defines a topology on X^*).

a) (X^*, Ω^*) is compact.

b) If X is locally compact, then X^* is Hausdorff.

c) A continuous map $f : X \rightarrow Y$ between Hausdorff topological spaces extends to a map $f^* : X^* \rightarrow Y^*$, which is continuous if f is proper (that is, the inverse image under f of every compact subset of Y is compact).

d) If X , and Y are locally compact Hausdorff spaces and if $f : X \rightarrow Y$ is proper and continuous, then f is a closed map.