

**METU - Mathematics Department
Graduate Preliminary Exam**

Topology

Duration : 3 hr.

Feb 9, 2007

1. Let τ be a Hausdorff topology on a set $X \neq \emptyset$ and for $A \subset X$, let \overline{A} denote the closure of A with respect to τ .

a) Prove that the family

$$\mathcal{B} = \{U - A : U \text{ is open, } \overline{A} \text{ is compact in } \tau\}$$

constitutes a basis for a new topology τ^* .

b) Prove that τ^* is Hausdorff.

c) Prove that $\tau = \tau^*$ if and only if all subsets of X which are compact in τ , are finite subsets.

2. Let X be an infinite set with the *finite complement topology* (ie. the collection of open sets is $\tau = \{A : X - A \text{ is finite, or } A = \emptyset\}$).

a) Prove that every subset of X is compact.

b) Prove that X is T_1 (ie. For every $x, y \in X$ with $x \neq y$, there are open sets U, V such that $x \in U - V$ and $y \in V - U$).

Is X Hausdorff? Is X metrizable?

c) If $X = \mathbb{R}$, find the closures and interiors of $(0, 1]$, $[2, 3]$, \mathbb{Z} .

3. Let X be a Hausdorff topological space and let $X^* = X \cup \{\infty\}$, where ∞ is an ideal point not in X . Consider the following collection Ω^* of subsets of X^*

(i) open sets in X

(ii) sets of the form $X^* - S$ where S is a compact subset of X .

Prove the following statements for the topological space (X^*, Ω^*) (**do not prove** that Ω^* defines a topology on X^*).

a) (X^*, Ω^*) is compact.

b) If X is locally compact, then X^* is Hausdorff.

c) A continuous map $f : X \rightarrow Y$ between Hausdorff topological spaces extends to a map $f^* : X^* \rightarrow Y^*$, which is continuous if f is proper (that is, the inverse image under f of every compact subset of Y is compact).

d) If Y is a locally compact Hausdorff space and if $f : X \rightarrow Y$ is proper, then f is a closed map.

4. Let (X, d) be a metric space. For a point x and for subspaces A, B in X , define $d(x, A) = \inf\{d(x, a) : a \in A\}$ and $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

(a) Prove that the map $f : X \rightarrow \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous.

(b) Prove that if A is compact, then there is a point $a_0 \in A$ such that $d(x, a_0) = d(x, A)$.

(c) Prove that if A and B are compact, then there are points $a_0 \in A$ and $b_0 \in B$ such that $d(a_0, b_0) = d(A, B)$.

(d) Prove that if A and B are compact and disjoint, then there are disjoint open sets U and V in X such that U contains A and V contains B .

(e) Show that the conclusion of (b) may not be true if A is not compact.