

**Graduate Preliminary Examination  
Numerical Analysis II**

**Duration: 3 hours**

1. Consider  $f(x) = (x - a)^n$  for some positive integer  $n$  and some real number  $a$

(a) Find the sequence  $\{x_i\}$  generated by the Newton and show that

$$x_{i+1} - a = \left(1 - \frac{1}{n}\right) (x_i - a)$$

(b) Find the order of convergence of the sequence  $\{x_i\}$ .

(c) Is this order compatible with the order of Newton's method? Give an explanation.

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2. Calculate a third order interpolating polynomial through the points  $(0, 0)$ ,  $(1, -2)$ ,  $(2, 0)$  and  $(3, 12)$  using Newton's Forward Divided Difference method. Give the table of differences, and compute the error of approximation of the resulting polynomial for  $x = 4$ .  
Would you get a different result using Newton's backward divided differences?
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3. Let  $\langle h, g \rangle = \int_a^b \omega(x)h(x)g(x)dx$  for  $h(x)$  and  $g(x)$  in  $C[a, b]$  and  $\omega(x)$  is a continuous positive weight function on  $(a, b)$ . Let  $\|h\| = \langle h, h \rangle^{1/2}$ .

(a) If  $f(x) \in C[a, b]$ , then the polynomial  $p_n^*(x) \in P_n$  which satisfies  $\|f - p_n^*\| \leq \|f - p\| \quad \forall p(x) \in P_n$  is given by

$$p_n^*(x) = \sum_{j=0}^n \langle f, p_j \rangle p_j(x)$$

where  $\{p_j(x)\}_{j=0}^n$  is the orthonormal set of polynomials generated by the Gram-Schmidt process with respect to the inner product given above ( $P_n$  is the set of  $n$ -th degree polynomials).

(b) Show that the remainder function  $(f(x) - p_n^*(x))$  is orthogonal to every polynomial in  $P_n$ .

(c) Show that

$$\|f - p_n^*\|^2 = \|f\|^2 - \sum_{j=0}^n \langle f, p_j \rangle^2 .$$

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4. Find an approximate formula for the evaluation of the integral

$$\int_0^1 f(x)x^{-1/2}dx$$

that is exact for all polynomial of degree one of the form

$$I(f) = c_1f(0) + c_2f(1).$$

\_\_\_\_\_ Determine the Peano kernel and the error term. \_\_\_\_\_