METU-MATHEMATICS DEPARTMENT
Graduate Preliminary Examinations

Topology

Duration: 3 hours

February 21, 2005

1. A collection $\{f_\alpha \mid \alpha \in A\}$ of functions on a space X (to spaces X_α) is said to separate points from closed sets in X iff whenever B is closed in X and $x \notin B$, then for some $\alpha \in A$, $f_\alpha(x) \notin \overline{f_\alpha(B)}$. Then, prove that a collection $\{f_\alpha \mid \alpha \in A\}$ of continuous functions on a topological space X separates points from closed sets in X if and only if the sets $f_\alpha^{-1}(V)$, for $\alpha \in A$ and V open in X_α, form a base for the topology on X.

2. Let X be a compact space and let $\{C_\alpha \mid \alpha \in A\}$ be a collection of closed sets, closed with respect to finite intersections. Let $C = \cap C_\alpha$ and suppose that $C \subset U$ with U open. Show that $C_\alpha \subset U$ for some $\alpha \in A$.

3. Let (X, d) be a metric space.

(a) Consider a connected set $A \subset X$ and a continuous function $f : X \to \mathbb{R}$. Given $\alpha, \beta \in f(A) \subset \mathbb{R}$ with $\alpha \leq \beta$ prove that for every $t \in \mathbb{R}$ with $\alpha \leq t \leq \beta$ there exists $a \in A$ such that $f(a) = t$.

(b) For any $B \subset X$, prove that the function $g : X \to \mathbb{R}$ defined by

$$g(x) = \operatorname{dist}(x, B) = \inf\{d(x, b) \mid b \in B\}$$

for all $x \in X$ is continuous.

(c) Let $\Omega \subset X$ be open, connected and relatively compact (i.e. its closure is compact). Consider a continuous surjective function $h : \Omega \to \Omega$. Prove that there exists $w \in \Omega$ such that

$$\operatorname{dist}(w, \operatorname{Bd}(\Omega)) = \operatorname{dist}(h(w), \operatorname{Bd}(\Omega)) \quad .$$

(Hint : Consider the point $w \in \overline{\Omega}$ at which $\operatorname{dist}(w, \operatorname{Bd}(\Omega))$ achieves its maximum.)

4. Let $X = \mathbb{R} \cup \{\infty\}$ with the topology with respect to which a subset of X not containing ∞ is open if it is open in the usual sense, a subset of X containing ∞ is open if its complement is the union of finitely many sequences convergent in the usual sense along with their limits.

(a) Prove that for any open set U in X the set $U \setminus \{\infty\}$ is open in $\mathbb{R}$ (Hint: the terms of a convergent sequence in $\mathbb{R}$ together with its limit is a closed set).

(b) Prove that $[0, 1] \subset X$ is compact.

(c) Prove that $[0, 1] \subset X$ is not closed. Is X Hausdorff?

(d)Prove that for every continuous $f : X \to X$ with $f(\infty) = \infty$, the set of fixed points $\{x \in X \mid f(x) = x\}$ is closed.